Mathematical Proofs

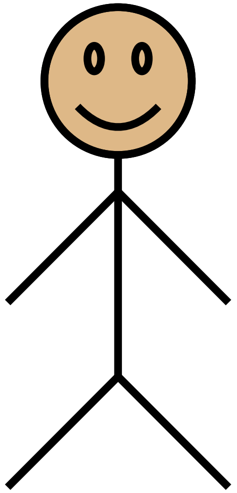
Outline for Today

- ***How to Write a Proof***
  - Synthesizing definitions, intuitions, and conventions.
- ***Proofs on Numbers***
  - Working with odd and even numbers.
- ***Universal and Existential Statements***
  - Two important classes of statements.
- ***Variable Ownership***
  - Who owns what?

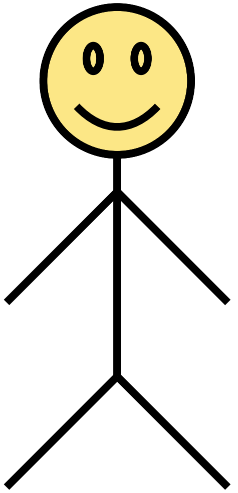
What is a Proof?

Proof as Dialog

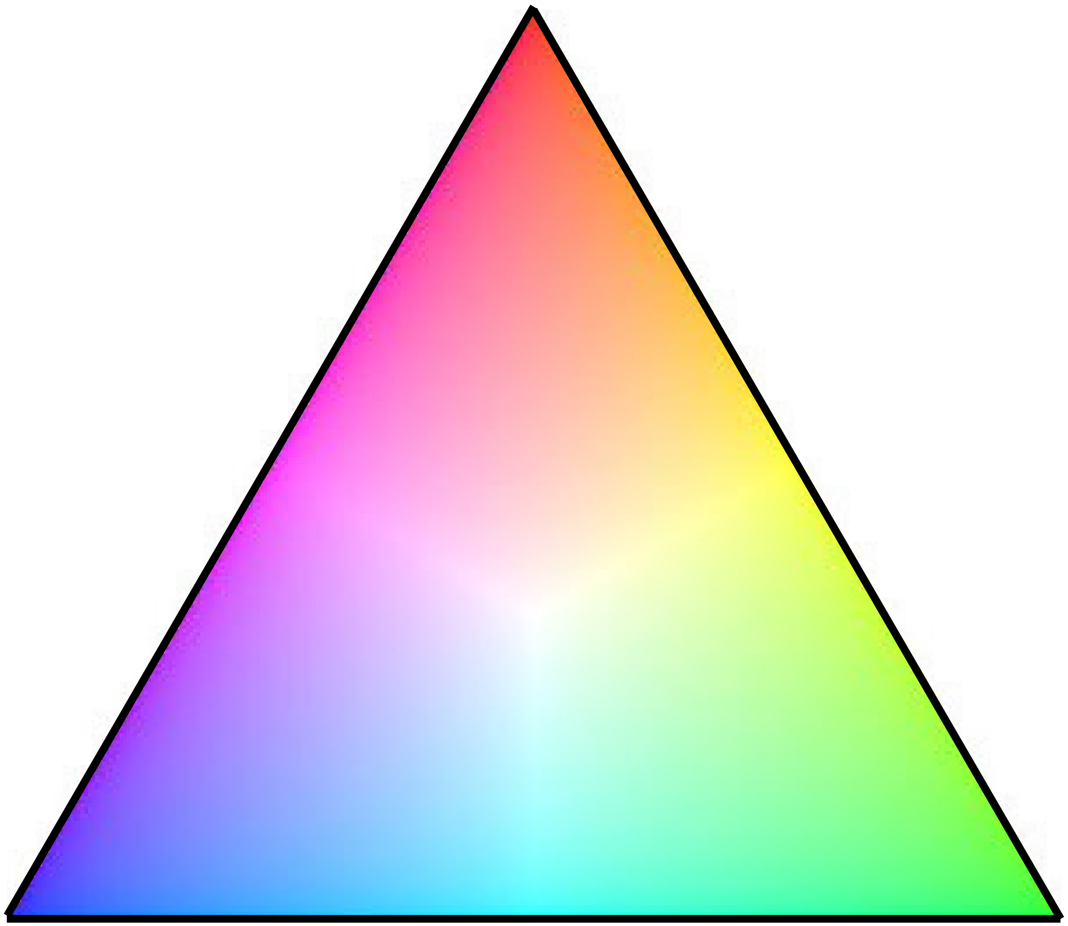
- A mathematical proof is a dialog between two parties: a ***proof writer*** and a ***proof reader***.
  - The ***proof writer*** knows a mathematical fact.
  - The ***proof reader*** is honest but skeptical.
- The proof writer's job is to take the reader on a journey from ignorance to understanding.



Proof Writer (You)



Proof Reader



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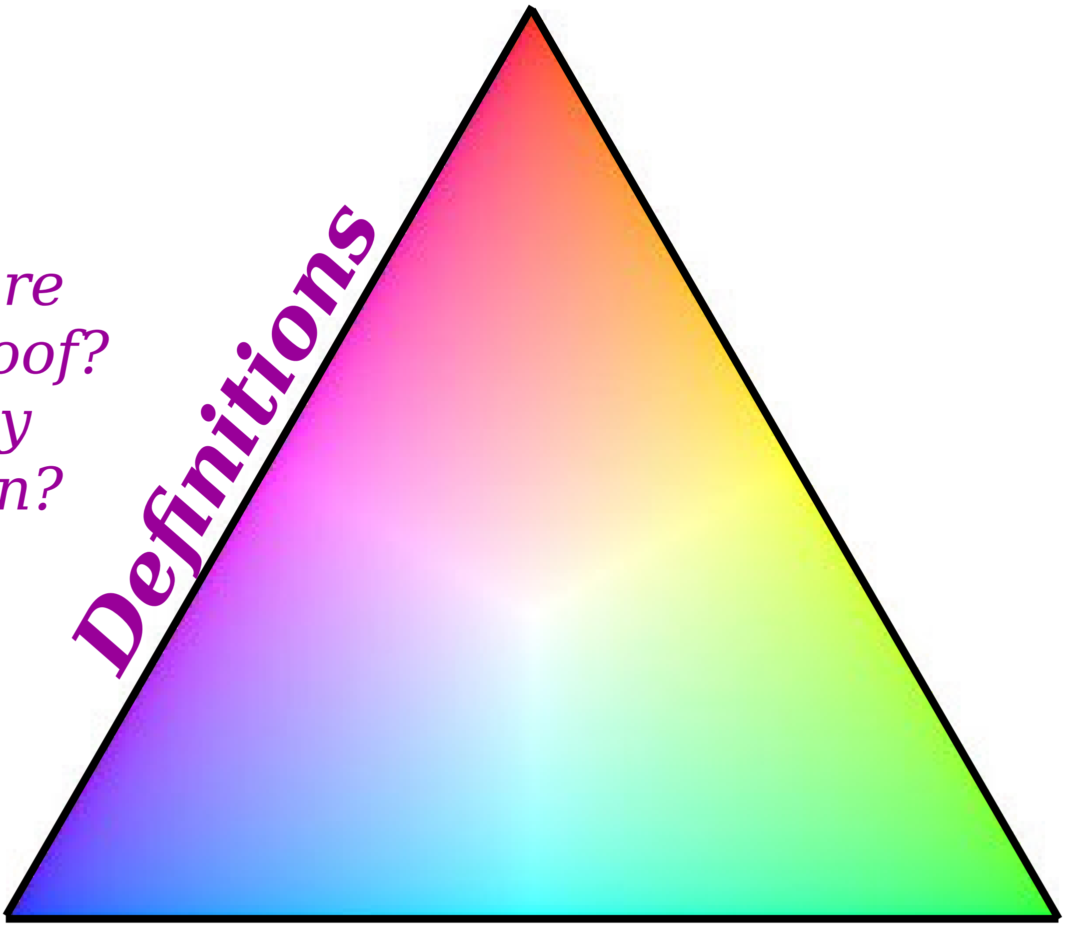

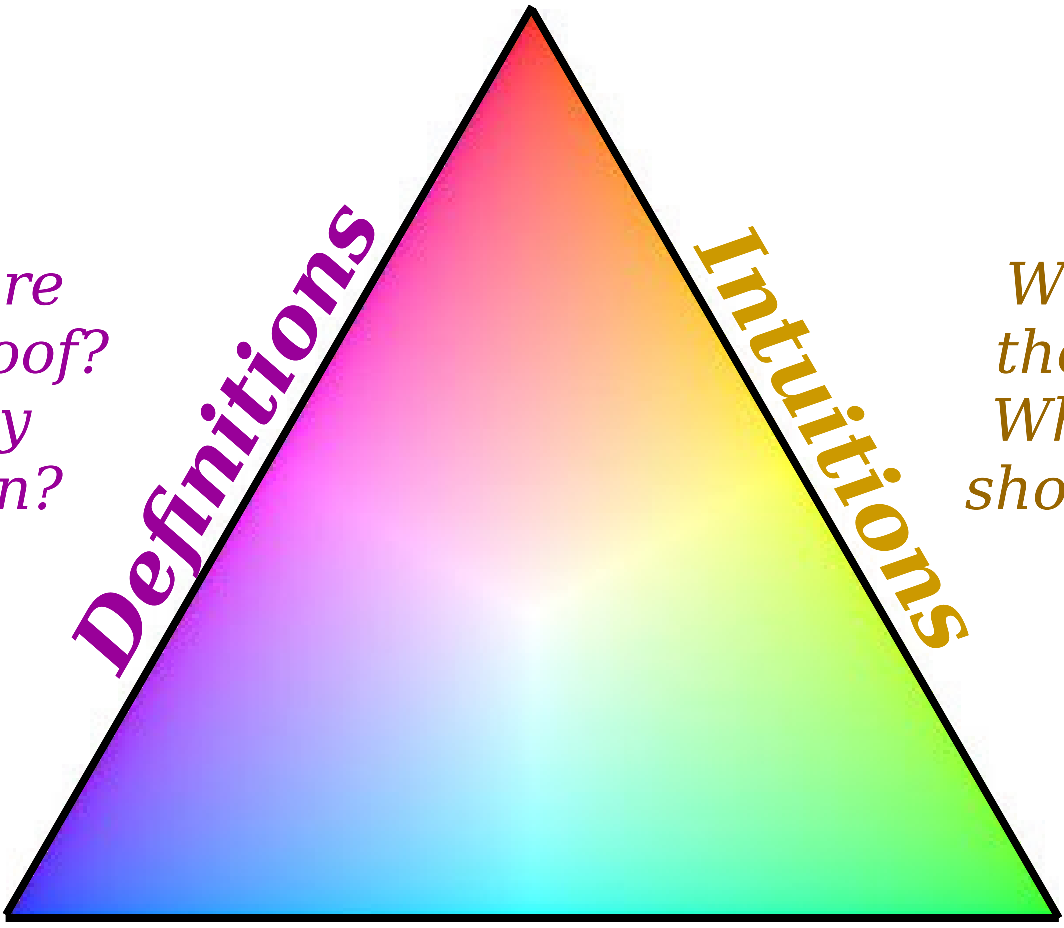

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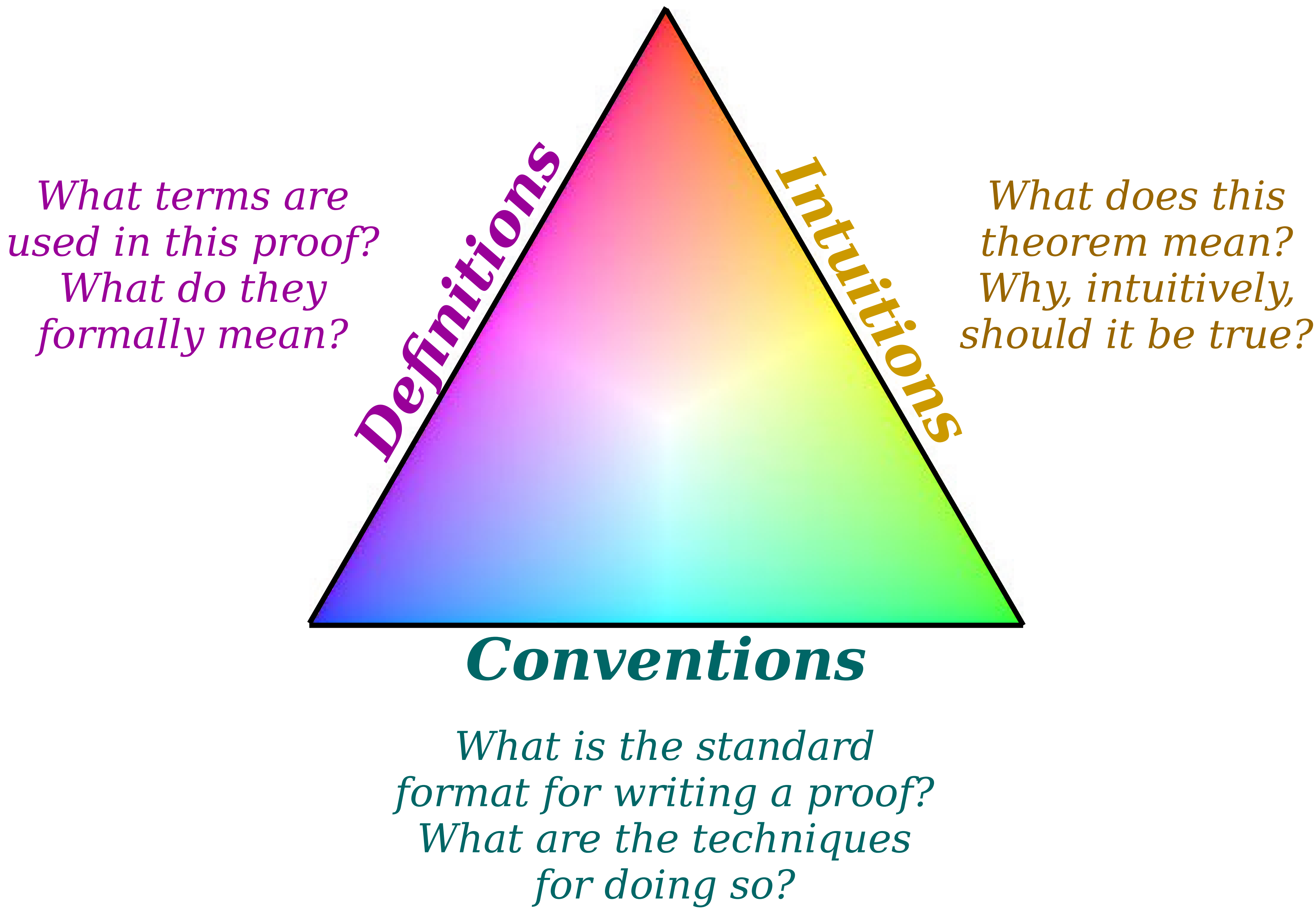
What does this theorem mean? Why, intuitively, should it be true?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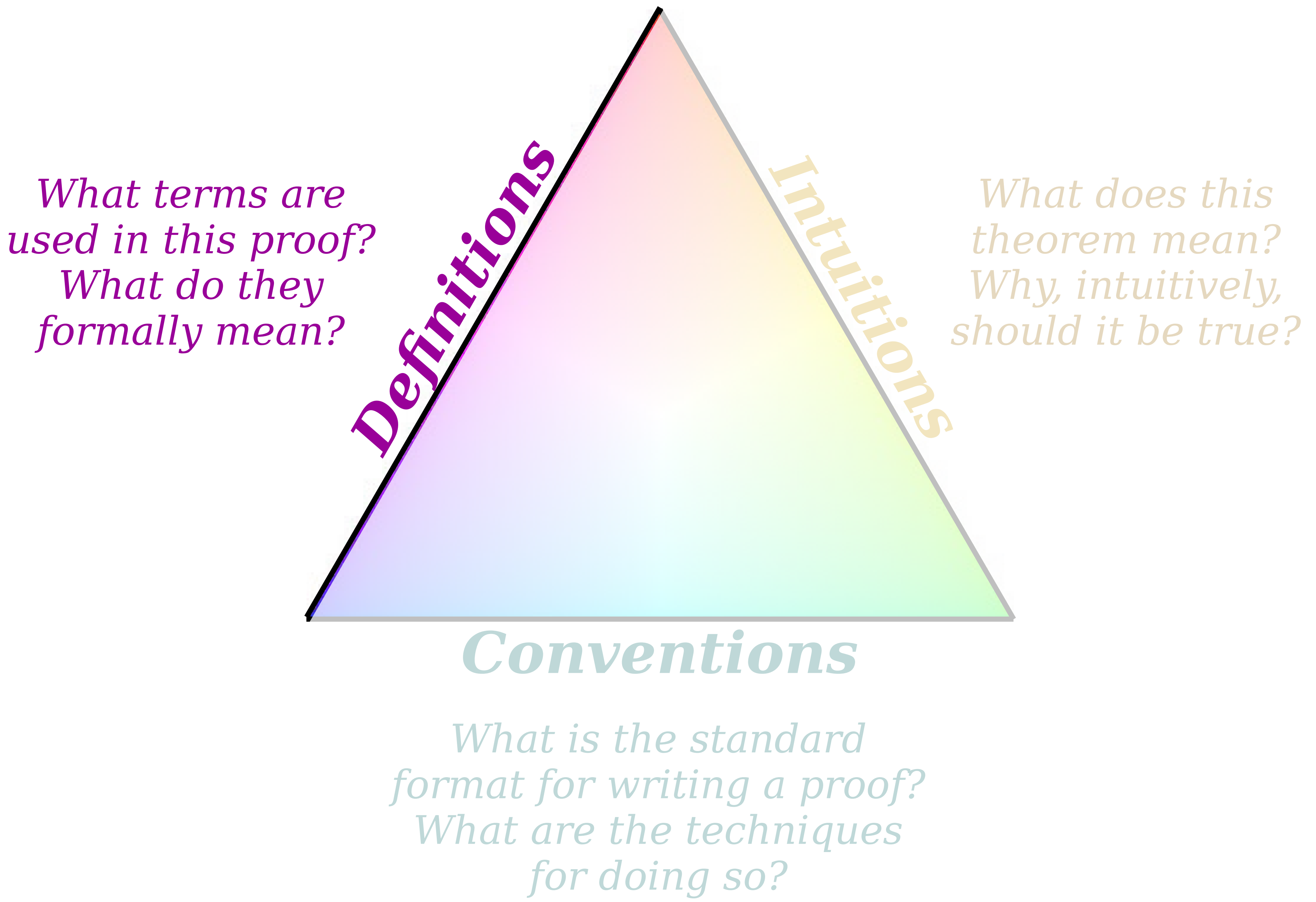
Writing our First Proof

Theorem: If n is an even integer, then n^2 is even.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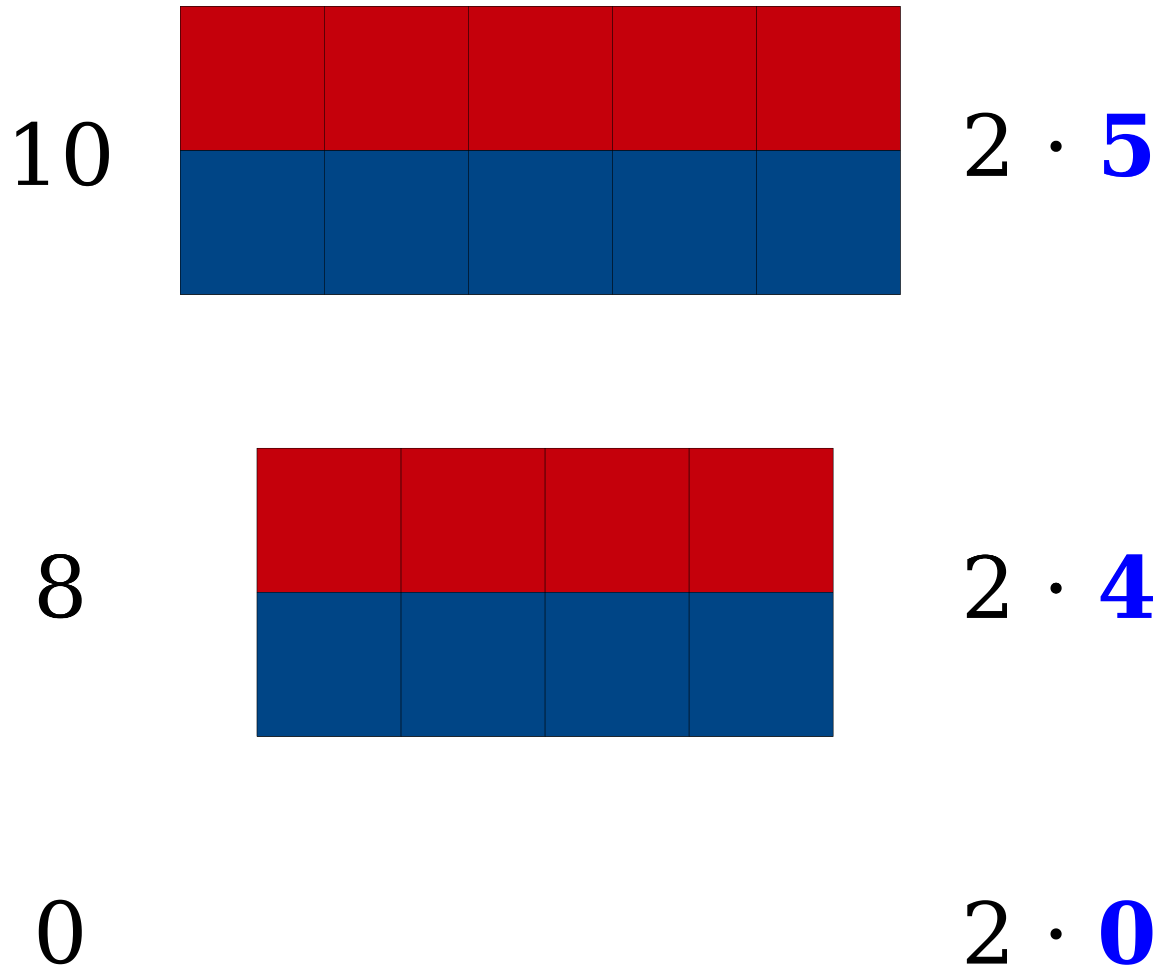

An integer n is called ***even*** if
there is an integer k where $n = 2k$.

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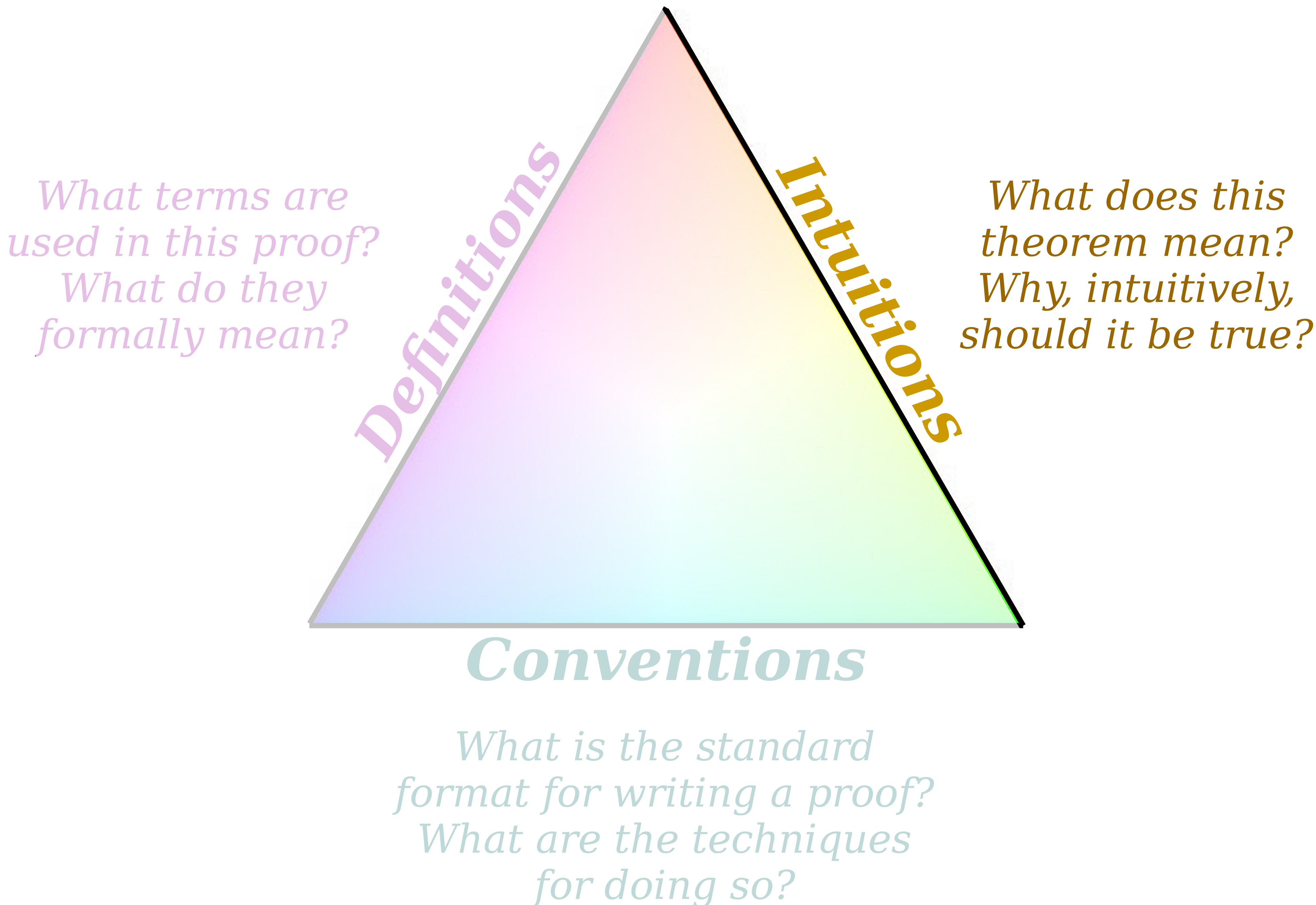
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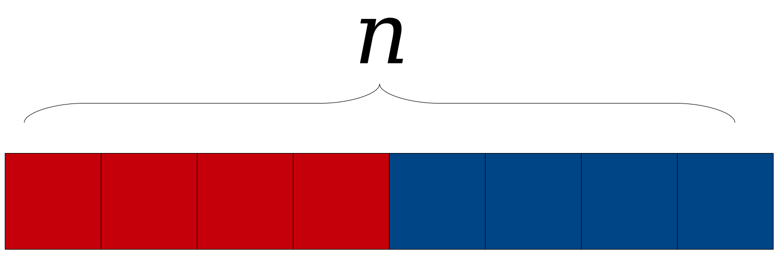
Let's Try Some Examples!

Theorem: If n is an even integer, then n^2 is even.

Let's Draw Some Pictures!



$n = 2k$

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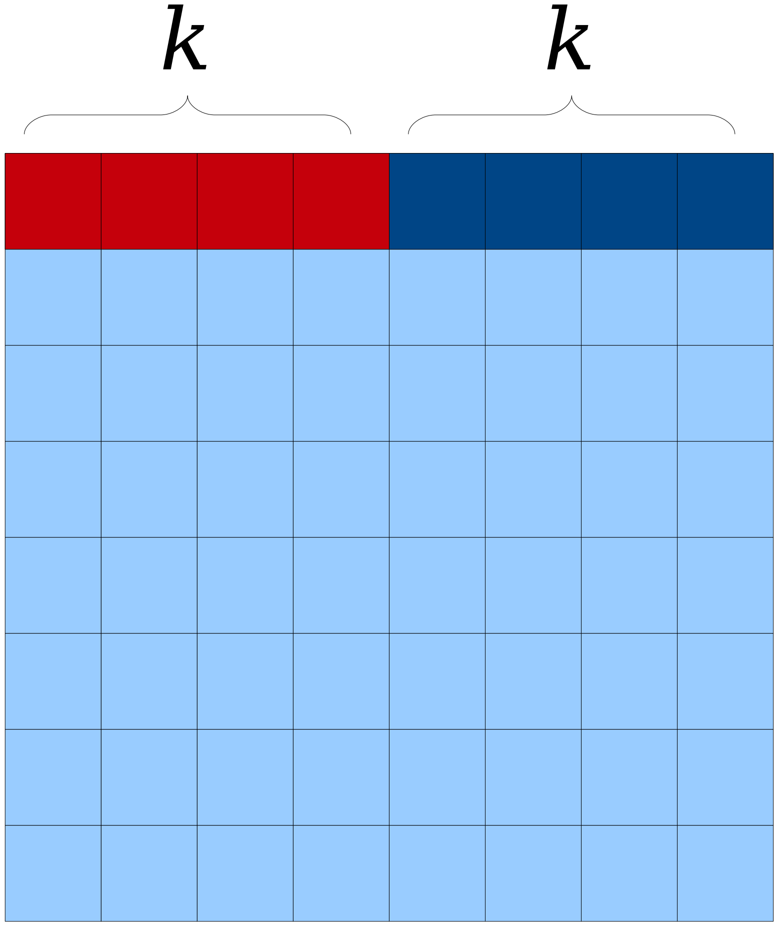

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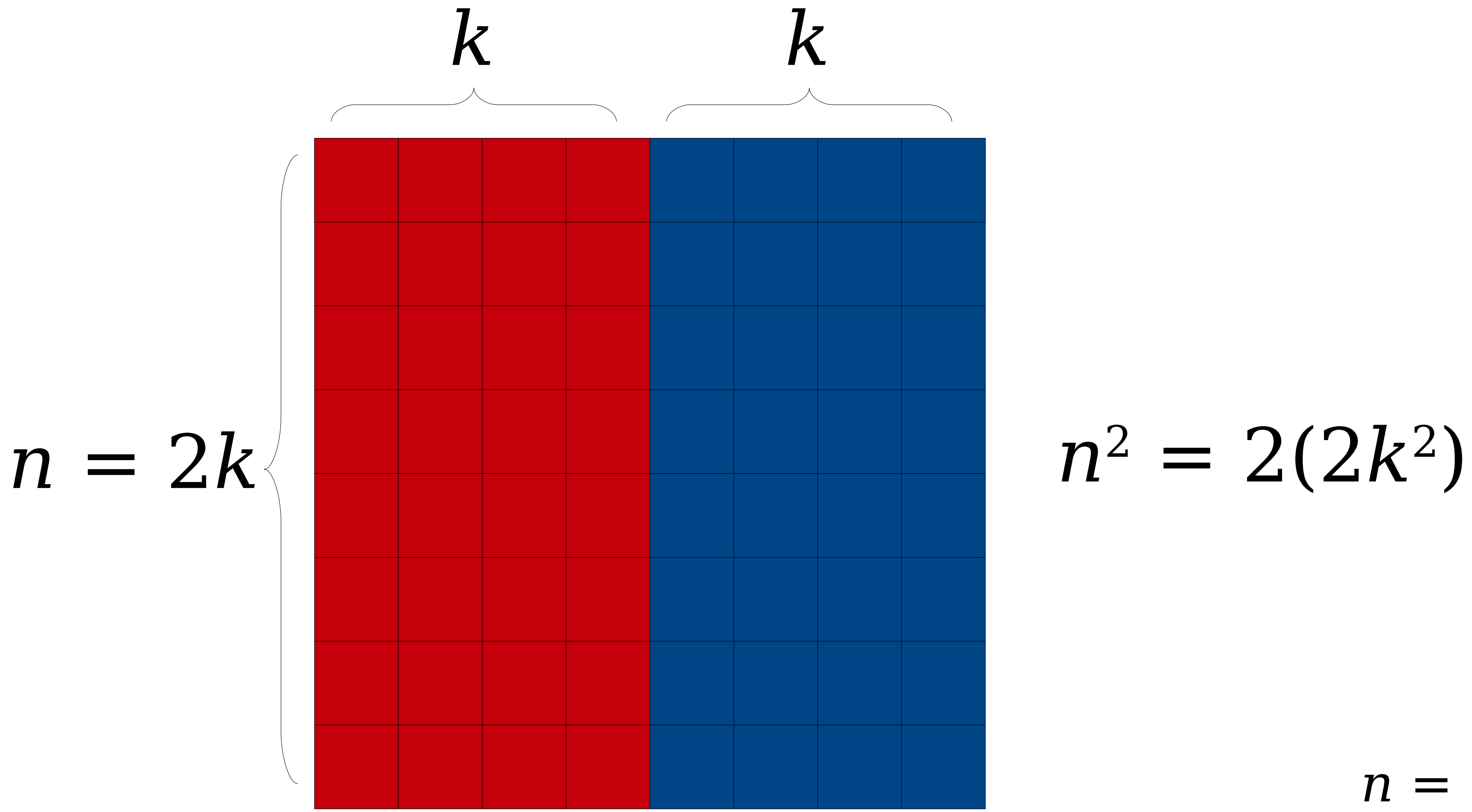

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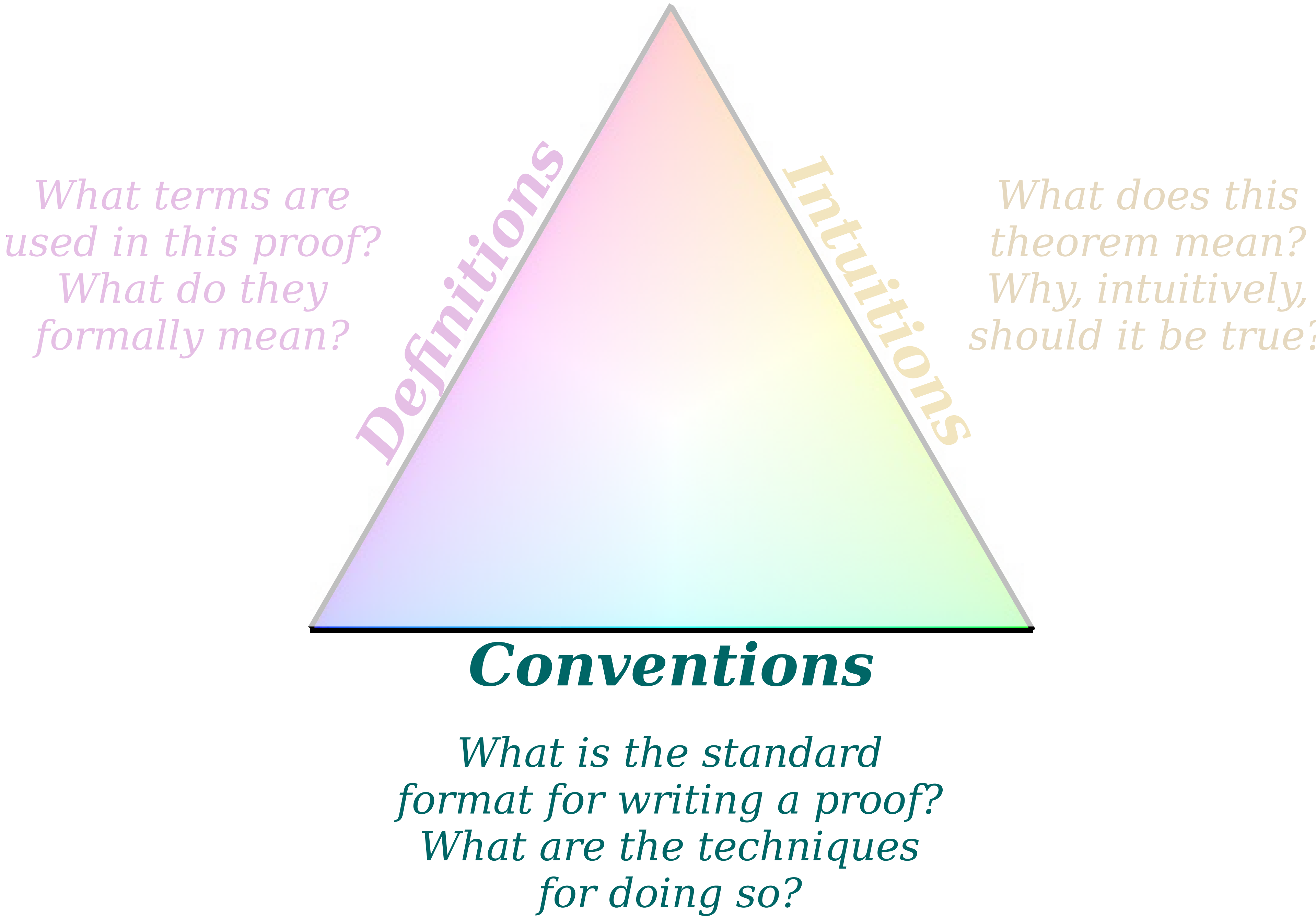
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$$n^2 = (2k)^2$$

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> This symbol means "end of proof"

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we assume ***P*** is true, then need to show that ***Q*** is true. Here, we're telling the reader where we're headed.

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> This is the definition of an even integer. We need to use this definition to make this proof rigorous.

From this, we see that there is an integer m (namely, $2k^2$) where $n^2 = 2m$. Therefore, n^2 is even, which is what we wanted to show. ■

Our First Proof!

Theorem: If

Proof: Assum
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Notice how we use the value of ***k*** that we obtained above. Giving names to quantities, allows us to manipulate them. This is similar to variables in programs.

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Since ... h that n ...

Our ultimate goal is to prove that $\boldsymbol{n^2}$ is even. This means that we need to find some $\boldsymbol{m}$ where $\boldsymbol{n^2 = 2m}$. Here, we're explicitly showing how we can do that.

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Hey, that's what we said we were going to do! We're done.

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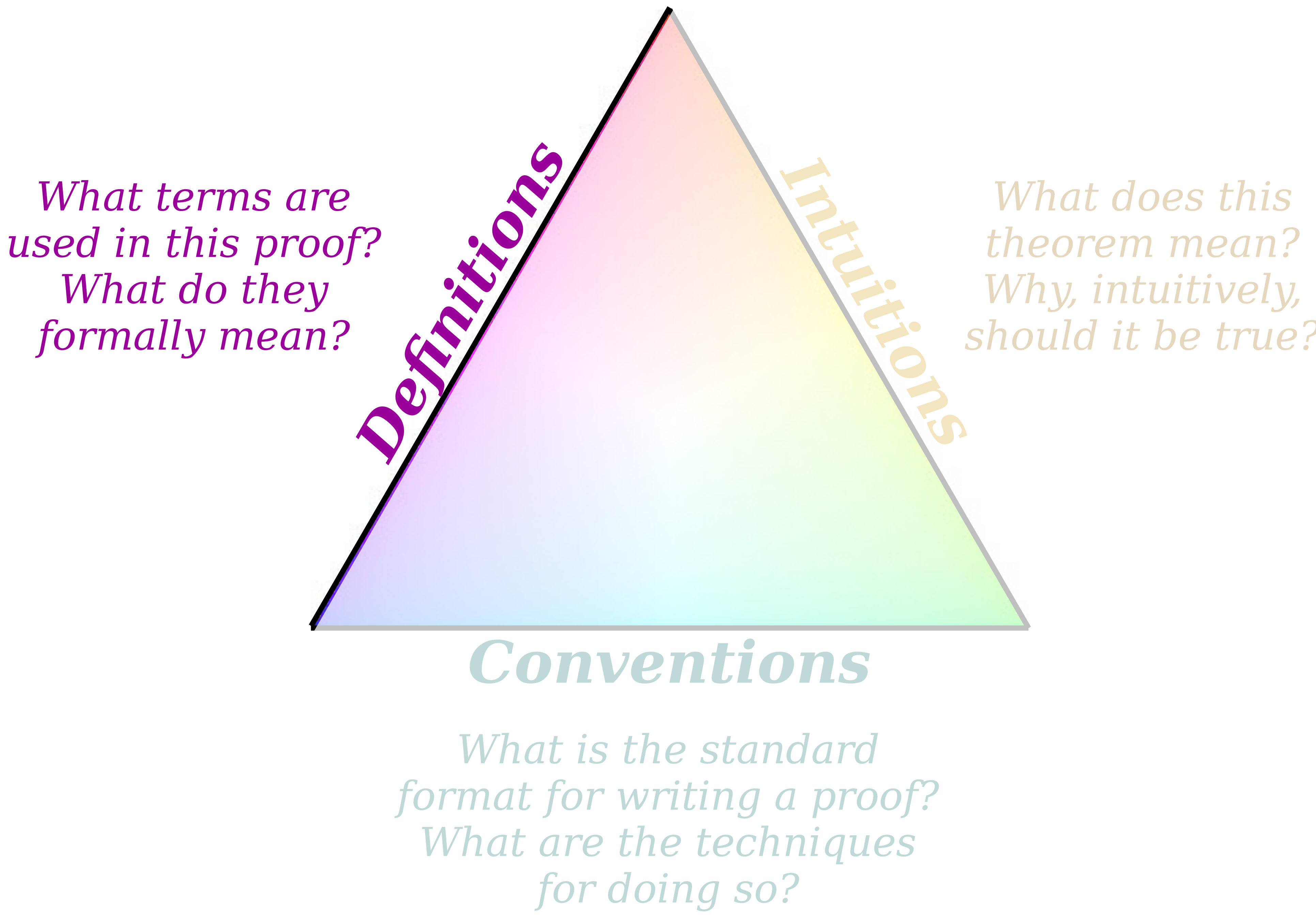
Our Next Proof

Theorem: For any integers m and n, if m and n are odd, then $m + n$ is even.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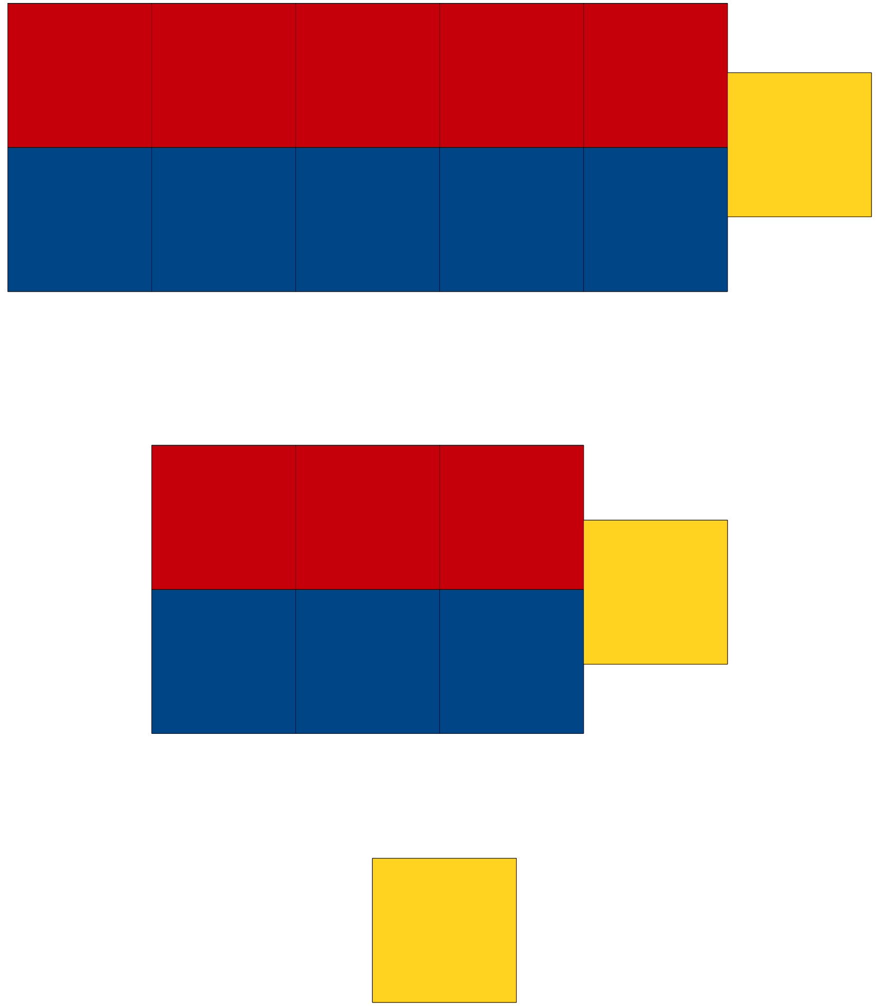
11 $2 \cdot \mathbf{5} + \mathbf{1}$

7 $2 \cdot \mathbf{3} + \mathbf{1}$

1 $2 \cdot \mathbf{0} + \mathbf{1}$

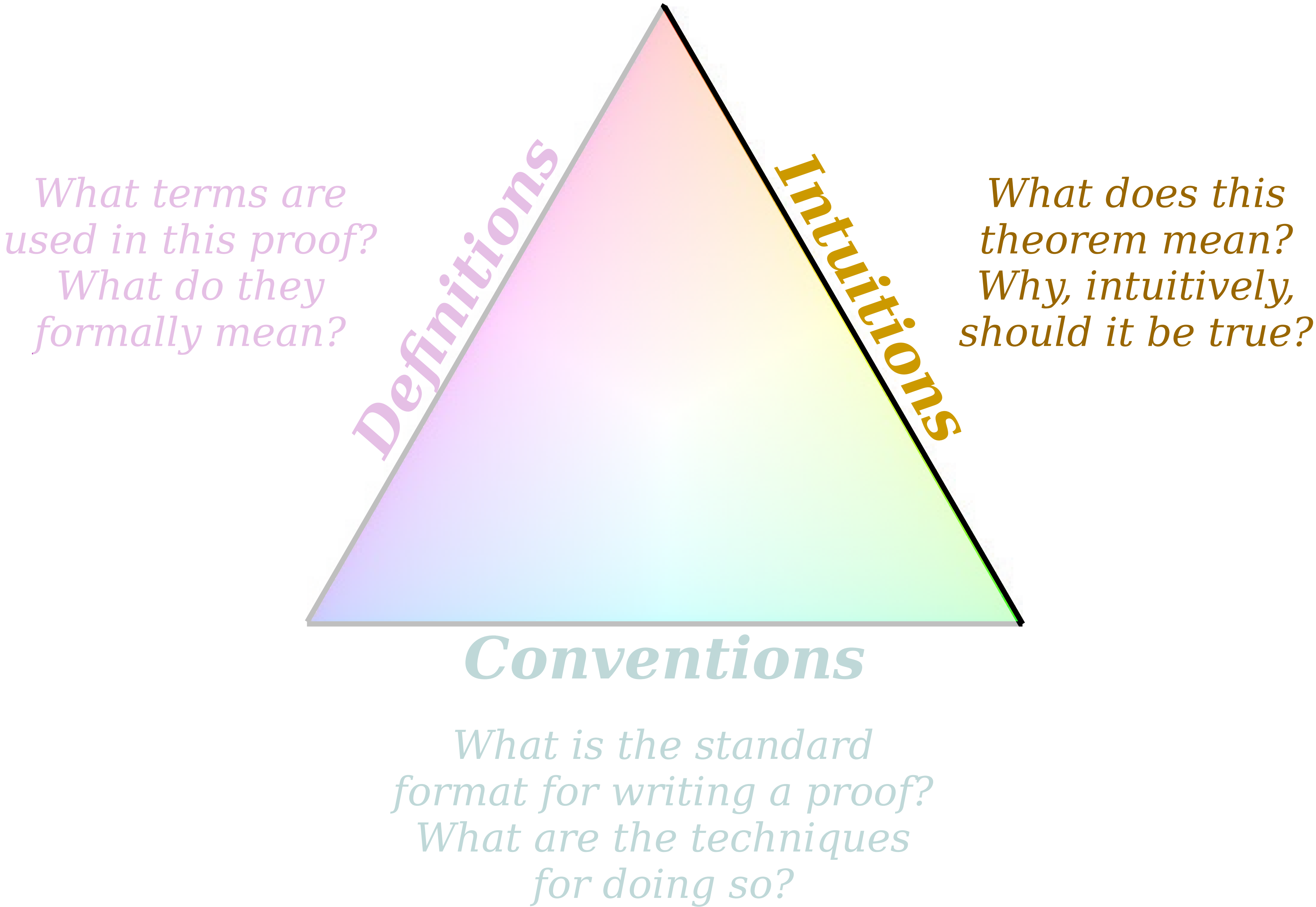


An integer n is called ***odd*** if
there is an integer k where $n = 2k+1$.

Going forward, we'll assume the following:

1. Every integer is either even or odd.
2. No integer is both even and odd.

Theorem: For any integers m and n, if m and n are odd, then $m + n$ is even.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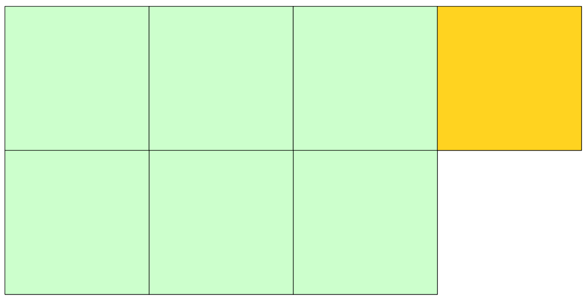

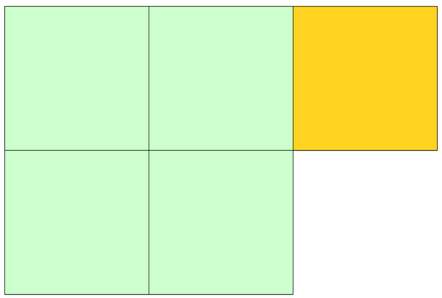

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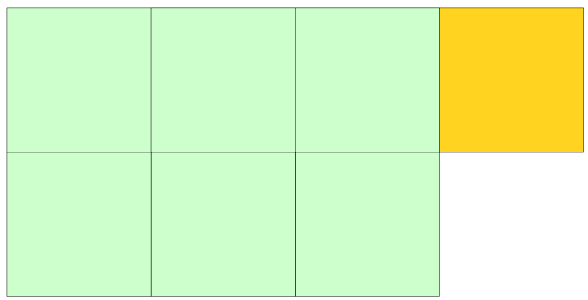

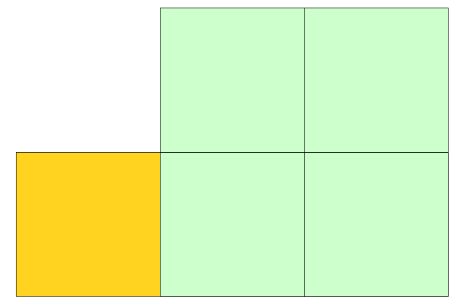

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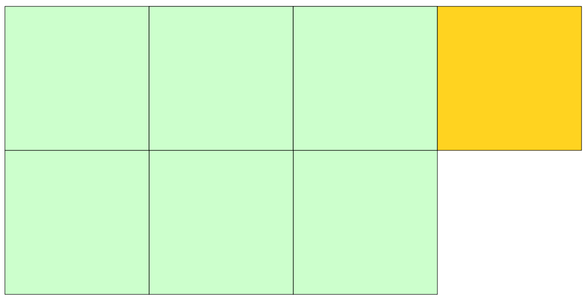

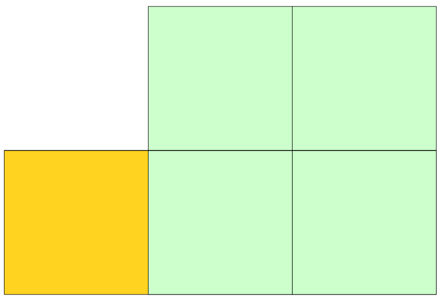

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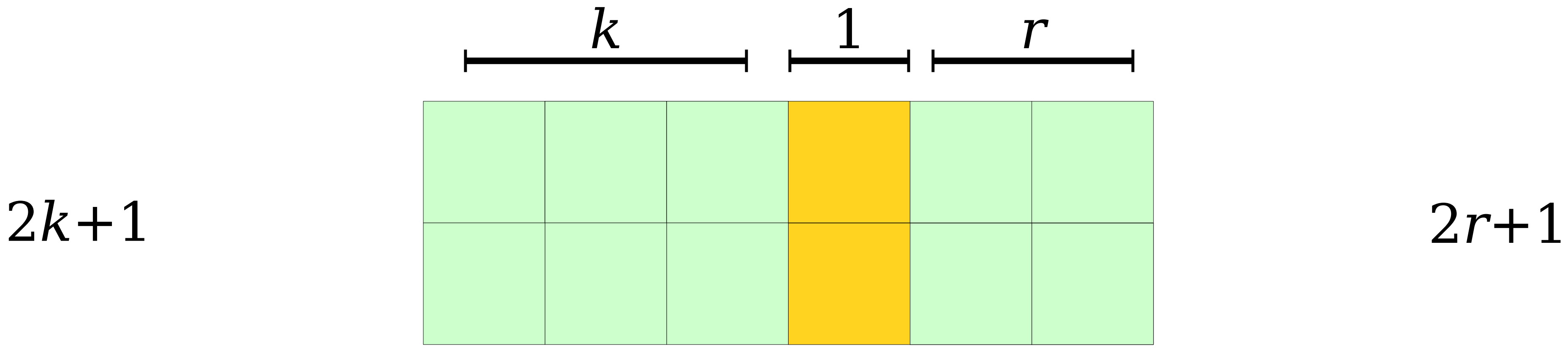

$$(2k+1) + (2r+1) = 2(k + r + 1)$$

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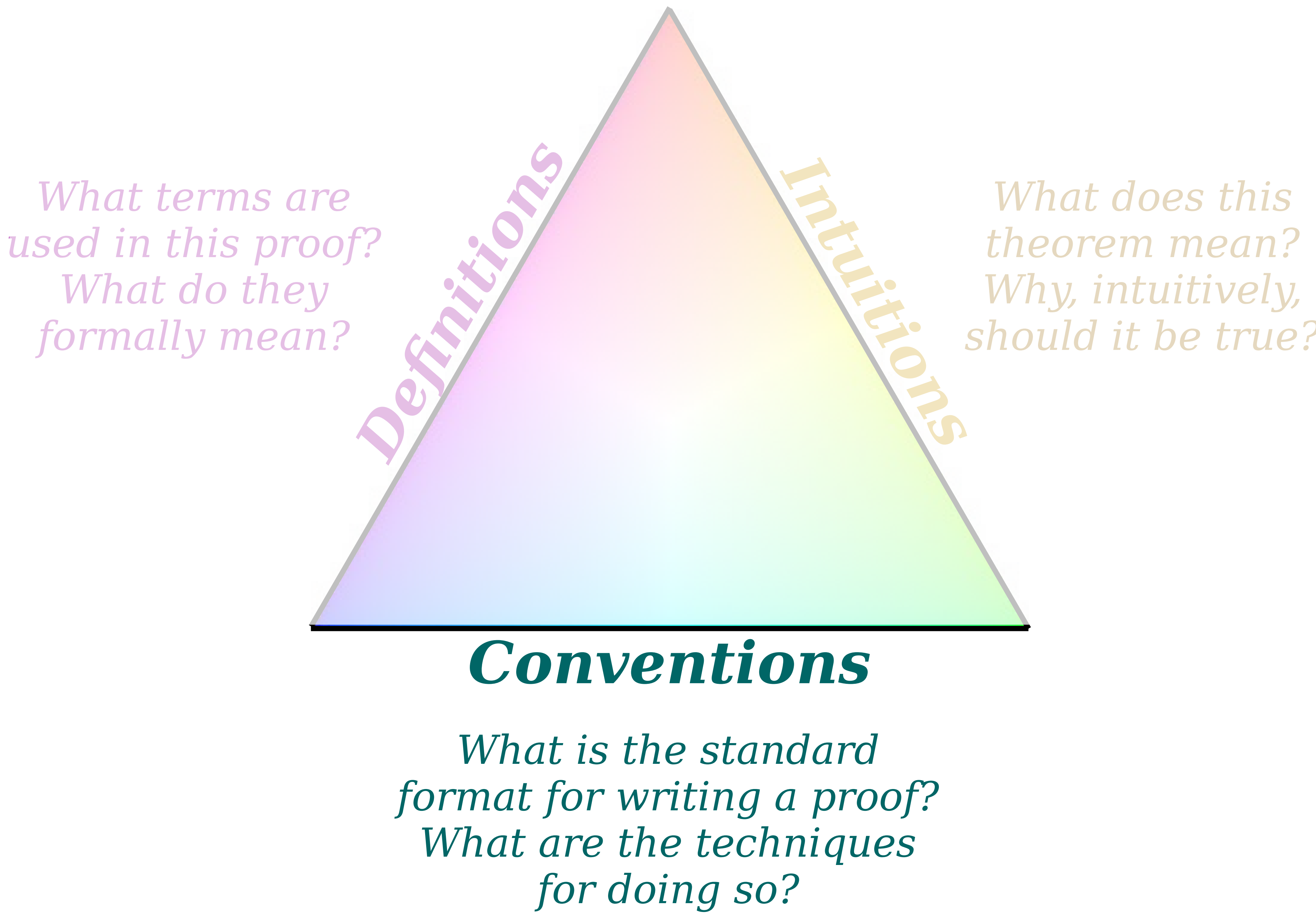
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We ask the reader to make an *arbitrary choice*. Rather than specifying what **m** and **n** are, we're signaling to the reader that they could, in principle, supply any choices of **m** and **n** that they'd like.

By letting the reader pick **m** and **n** arbitrarily, anything we prove about **m** and **n** will generalize to all possible choices for those values.

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start by asking the reader to assume that ***P*** is true.

$$= 2k + 2r + 2$$

$$= 2(k + r + 1). \qquad (3)$$

Equation (3) tells us that there is an integer s (namely, $k + r + 1$) such that $m + n = 2s$. Therefore, we see that $m + n$ is even, as required. ■

Theorem: For any integers m and n, if m and n are odd, then $m + n$ is even.

Proof: Consider any arbitrary integers m and n where m and n are odd. We need to show that $m + n$ is even.

Since m i

Similarly, ch that

By adding

To prove a statement of the form

"If *P* is true, then *Q* is true,"

after assuming ***P*** is true, you need to show that ***Q*** is true.

$= 2k + 2r + 2$

$= 2(k + r + 1).$ (3)

Equation (3) tells us that there is an integer s (namely, $k + r + 1$) such that $m + n = 2s$. Therefore, we see that $m + n$ is even, as required. ■

Theorem: For any integers m and n, if m and n are odd, then $m + n$ is even.

Numbering these equalities lets us refer back to them later on, making the flow of the proof a bit easier to understand.

Proof: Consider any

odd. We need to s

Since m is odd, w

$$m = 2k + 1. \qquad (1)$$

Similarly, because n is odd there must be some integer r such that

$$n = 2r + 1. \qquad (2)$$

By adding equations (1) and (2) we learn that

$$\begin{aligned} m + n &= 2k + 1 + 2r + 1 \\ &= 2k + 2r + 2 \\ &= 2(k + r + 1). \qquad (3) \end{aligned}$$

Equation (3) tells us that there is an integer s (namely, $k + r + 1$) such that $m + n = 2s$. Therefore, we see that $m + n$ is even, as required. ■

Theorem: For any integers m and n, if m and n are odd, then $m + n$ is even.

Proof: Consider any arbitrary integers m and n where m and n are odd. We need to show that $m + n$ is even.

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(2)

[...]arn that

[...] $2r + 1$

[...] $+ 2$

[...] $+ 1)$. (3)

[...]nteger s (namely, $k + r + 1$) [...]see that $m + n$ is even, as

This is a complete sentence! Proofs are expected to be written in complete sentences, so you'll often use punctuation at the end of formulas.

We recommend using the "mugga mugga" test – if you read a proof and replace all the mathematical notation with "mugga mugga," what comes back should be a valid sentence.

Theorem: For any integers m and n, if m and n are odd, then $m + n$ is even.

Proof: Consider any arbitrary integers m and n where m and n are odd. We need to show that $m + n$ is even.

Since m is odd, we know that there is an integer k where

$$m = 2k + 1. \tag{1}$$

Similarly, because n is odd there must be some integer r such that

$$n = 2r + 1. \tag{2}$$

By adding equations (1) and (2) we learn that

$$\begin{aligned} m + n &= 2k + 1 + 2r + 1 \\ &= 2k + 2r + 2 \\ &= 2(k + r + 1). \end{aligned} \tag{3}$$

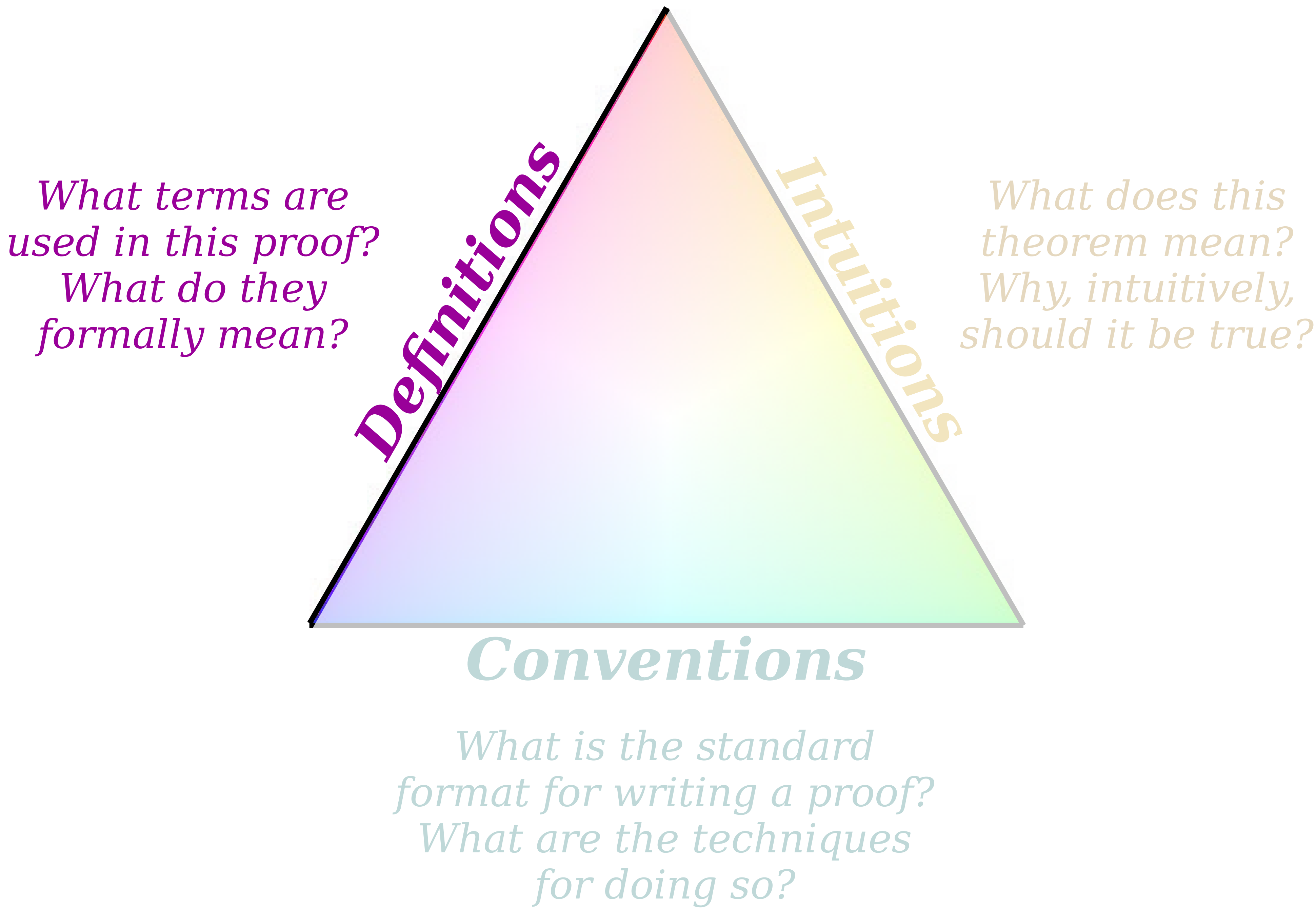
Equation (3) tells us that there is an integer s (namely, $k + r + 1$) such that $m + n = 2s$. Therefore, we see that $m + n$ is even, as required. ■

Some Little Exercises

- Here’s a list of other theorems that are true about odd and even numbers:
  - ***Theorem:*** The sum and difference of any two even numbers is even.
  - ***Theorem:*** The sum and difference of an odd number and an even number is odd.
  - ***Theorem:*** The product of any integer and an even number is even.
  - ***Theorem:*** The product of any two odd numbers is odd.
- Going forward, we’ll just take these results for granted. Feel free to use them in the problem sets.
- If you’d like to practice the techniques from today, try your hand at proving these results!

Universal and Existential Statements

Theorem: For any odd integer n, there exist integers r and s where $r^2 - s^2 = n$.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Theorem: For any odd integer n,
there exist integers r and s where $r^2 - s^2 = n$.

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This result is true for every possible choice of odd integer n. It'll work for $n = 1$, $n = 137$, $n = 103$, etc.

Theorem: For any odd integer n, there exist integers r and s where $r^2 - s^2 = n$.

We aren't saying this is true for every choice of r and s. Rather, we're saying that *somewhere out there* are choices of r and s where this works.

Universal vs. Existential Statements

- A ***universally-quantified statement*** is a statement of the form

For all *x*, `[some-property]` holds for *x*.

- We've seen how to prove these statements.
- An ***existentially-quantified statement*** is a statement of the form

There is some *x* where `[some-property]` holds for *x*.

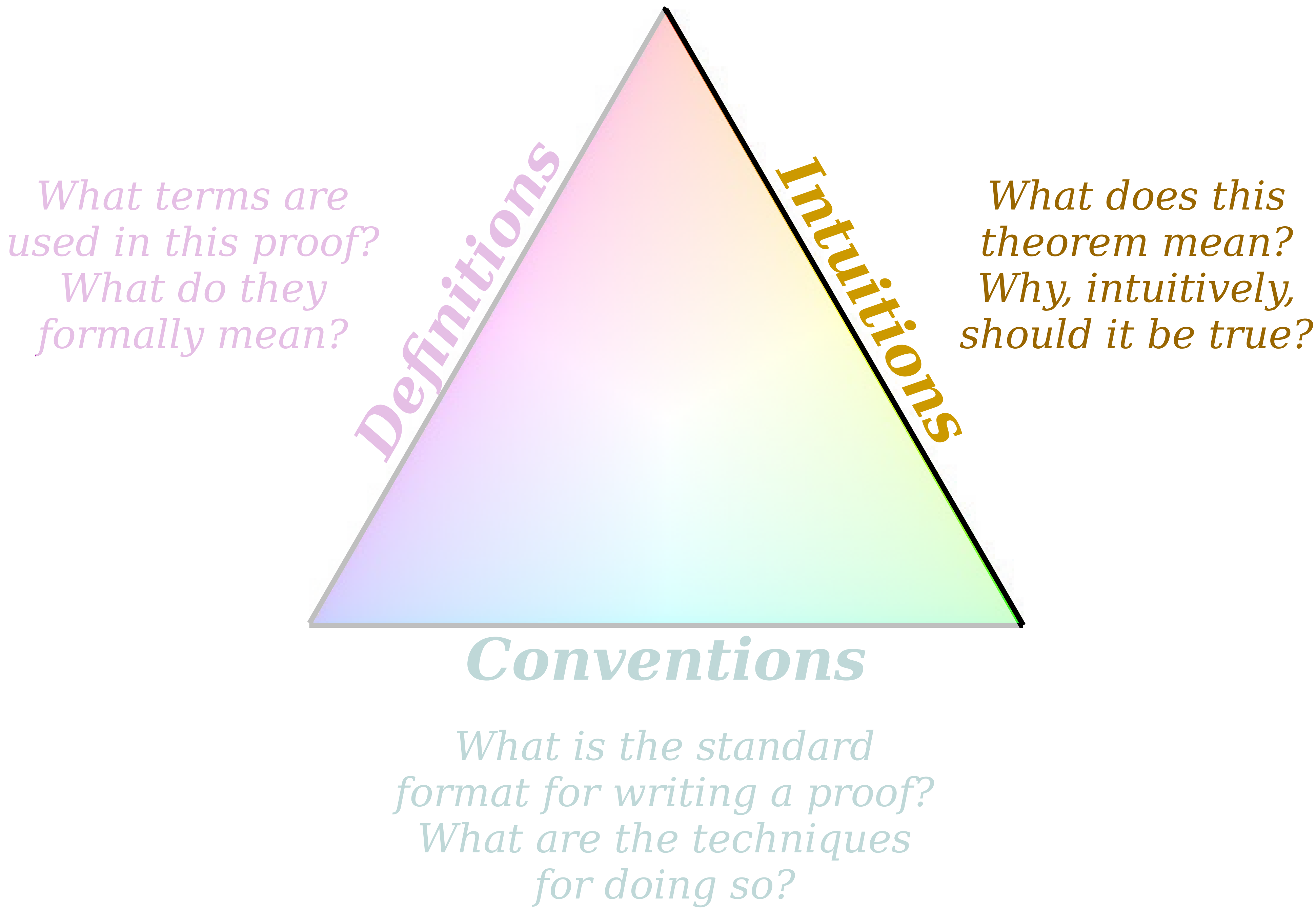
- How do you prove an existentially-quantified statement?

Proving an Existential Statement

- Over the course of the quarter, we will see several different ways to prove an existentially-quantified statement of the form

  There is an *x* where `[some-property]` holds for *x*.

- ***Simplest approach:*** Search far and wide, find an *x* that has the right property, then show why your choice is correct.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Let’s Try Some Examples!

$1 = \_\_\_^2 - \_\_\_^2$

$3 = \_\_\_^2 - \_\_\_^2$

$5 = \_\_\_^2 - \_\_\_^2$

$7 = \_\_\_^2 - \_\_\_^2$

$9 = \_\_\_^2 - \_\_\_^2$

Theorem: For any odd integer n, there exist integers r and s where $r^2 - s^2 = n$.

Let’s Try Some Examples!

$$1 = \mathbf{1}^2 - \mathbf{0}^2$$

$$3 = \mathbf{2}^2 - \mathbf{1}^2$$

$$5 = \mathbf{3}^2 - \mathbf{2}^2$$

$$7 = \mathbf{4}^2 - \mathbf{3}^2$$

$$9 = \mathbf{5}^2 - \mathbf{4}^2$$

Theorem: For any odd integer n, there exist integers r and s where $r^2 - s^2 = n$.

Let's Try Some Examples!

$$1 = 2 \cdot \_ + 1 = \mathbf{1}^2 - \mathbf{0}^2$$

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$$\mathbf{3} + 1 = \mathbf{4}^2 - \mathbf{3}^2$$

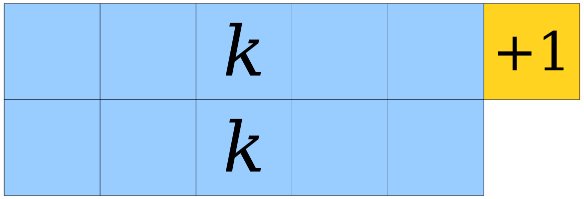
$$\mathbf{4} + 1 = \mathbf{5}^2 - \mathbf{4}^2$$

Educated Guess:

$2k + 1 = (k+1)^2 - k^2.$

Theorem: For any odd integer n, there exist integers r and s where $r^2 - s^2 = n$.

Let's Draw Some Pictures!



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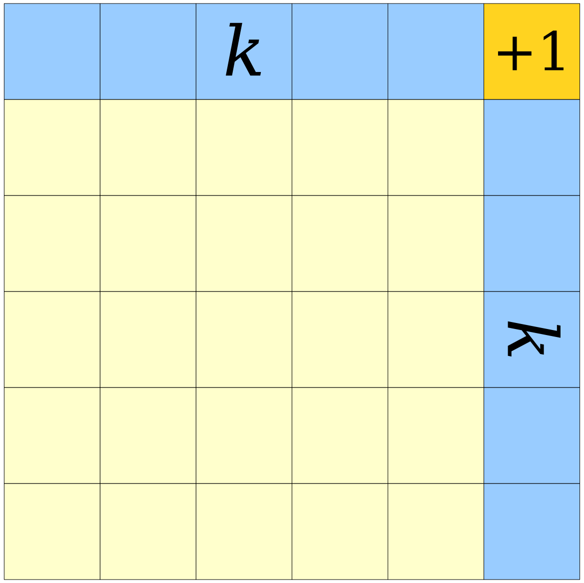
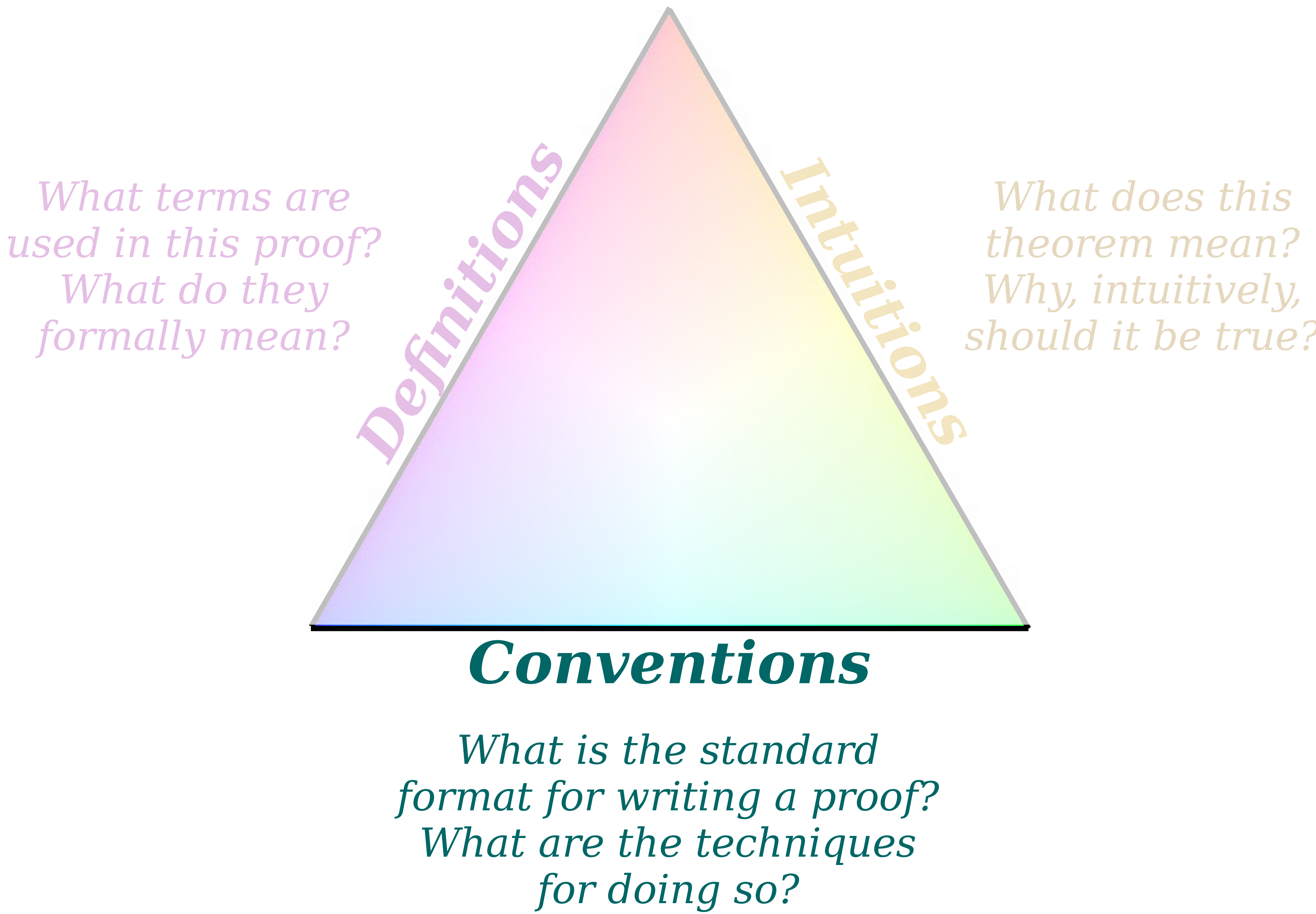


$$(k+1)^2 - k^2 = 2k+1$$

Theorem: For any odd integer n, there exist integers r and s where $r^2 - s^2 = n$.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Proof:

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Proof: Let n be an arbitrary odd integer.

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$$r^2 - s^2 \quad = \quad (k+1)^2 - k^2$$

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This means that $r^2 - s^2 = n$, which is what we needed to show.

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Proof: Let n be an arbitrary odd integer. We will show that there exist integers r and s where $r^2 - s^2 = n$.

Since r … there
$n = 2k$ … e see
that

We ask the reader to make an *arbitrary choice*. Rather than specifying what ***n*** is, we're signaling to the reader that they could, in principle, supply any choice ***n*** that they'd like.

$$= \quad 2k + 1$$

$$= \quad n.$$

This means that $r^2 - s^2 = n$, which is what we needed to show. ■

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Proof: Let n be an arbitrary odd integer. We will show that there exist integers r and s where $r^2 - s^2 = n$.

Since n is odd, we know th… re $n = 2k + 1$. Now, let $r = k$… ee that

> As always, it's helpful to write out what we need to demonstrate with the rest of the proof.

$$
\begin{aligned}
r^2 - s^2 &= (k+ \ldots \\
&= k^2 + 2k + 1 - k^2 \\
&= 2k + 1 \\
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Since n is odd, we know there is an integer k where $n = 2k + 1$. Now, let $r = k+1$ and $s = k$. Then we see that

$$r^2 - s^2 = (k+1)^2 \ldots$$
$$= k^2 + 2 \ldots$$
$$= 2k + 1 \ldots$$
$$= n.$$

This means that $r^2 - s^2 = n$, w… to show. ■

We're trying to prove an existential statement. The easiest way to do that is to just give concrete choices of the objects being sought out.

Theorem: For any odd integer n, there exist integers r and s where $r^2 - s^2 = n$.

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This means that $r^2 - s^2 = n$, which is what we needed to show. ■

Check the appendix to this slide deck for more about who gets to choose values.

Time-Out for Announcements!

Working in Pairs

- Starting with Problem Set One, you are allowed to work either individually or in pairs.
  - Each pair should make a single joint submission.
- We have advice about how to work effectively in pairs up on the course website – check the “Guide to Partners.”
- Want to work in a pair, but don’t know who to work with? Fill out ***this Google form*** and we’ll connect you with a partner on Friday.

Problem Set 0

- Problem Set 0 is due this Friday at 1:00PM.
  - (It needs to be completed individually.)
- Need help getting Qt Creator installed? There’s a Qt Creator help session running tonight, 6PM – 8PM, in Durand 353.

CS103 ACE

- Reminder: There’s an optional companion course, CS103 ACE, that runs in parallel with CS103.
- CS103 ACE meets Tuesdays 3:00 – 4:50PM and provides additional practice with the course material in a small group setting.
- The first course meeting is next Tuesday.
- Interested? Apply online using ***this link***.
- The CS103 ACE materials are available to everyone. You can pull them up ***here***.

Preview: Lecture Participation

- 5% of your course grade is allocated to lecture participation, which starts next Wednesday.
- We'll use PollEV to ask questions in lecture to help solidify understanding.
- You'll get participation credit for the day if you answer those questions, regardless of whether your answers are right.
- You get three free missed lectures without penalty.
- If you aren't able to make it to lecture, or would prefer to watch asynchronously, you can opt to count your final exam score in place of your participation grade. (We'll send out a form in Week 4 you can use to opt out.)

Outdoor Activities

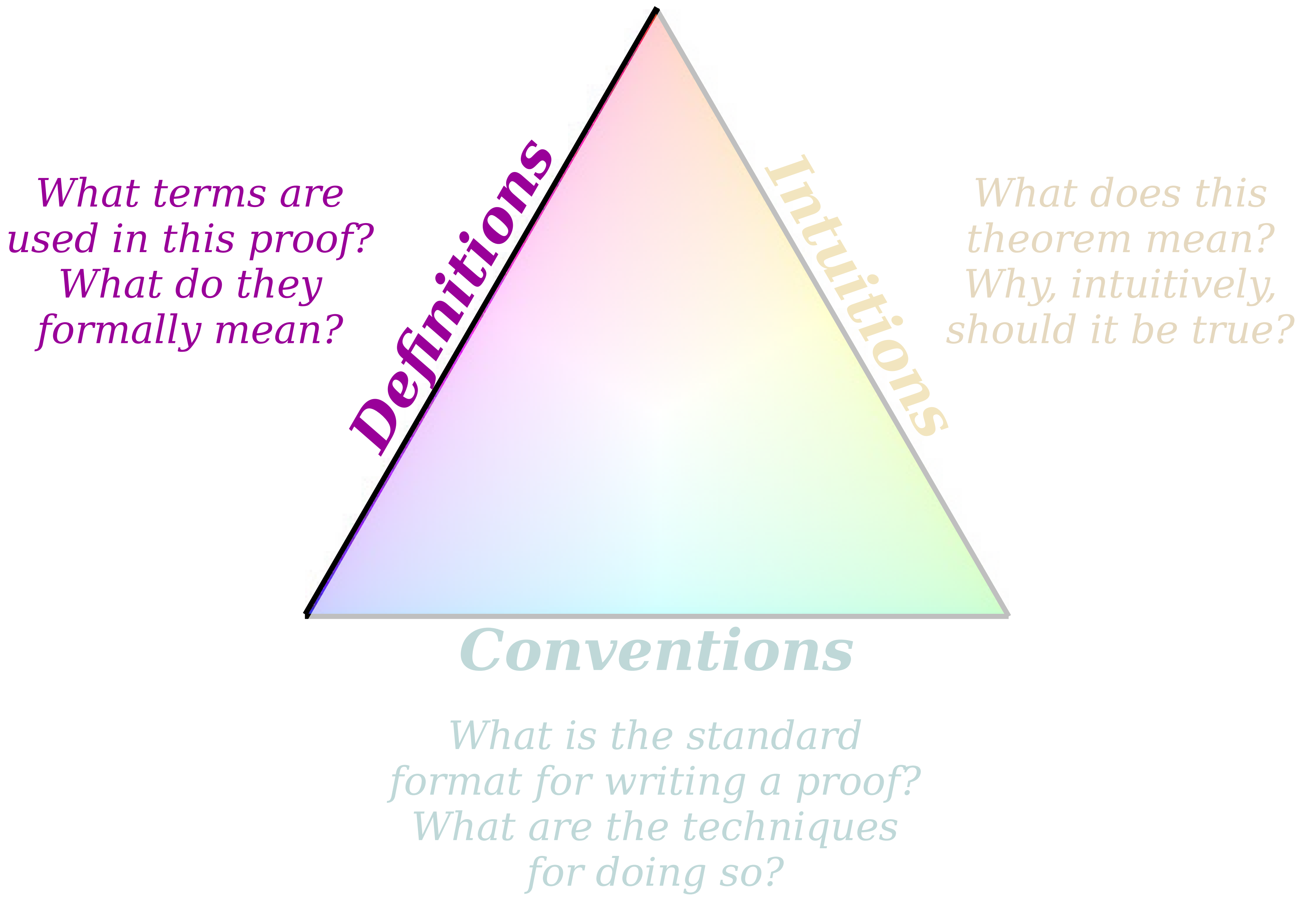
- You're less than fifty miles from grassy mountains, redwood forests, Pacific coastline, beautiful wetlands, and more.
- Want to explore the area to see what it has to offer? Check out our (unofficial) Outdoor Activities Guide.

  https://cs103.stanford.edu/outdoor_activities

- A sampler of what to check out:
  - Drive to the observatory in the mountains near San Jose and take in the views.
  - Visit a beach with an enormous colony of elephant seals.
  - Walk in redwood forests and pick your own bay leaves.
  - Grab cheap, high-quality food from unassuming strip malls.

Back to CS103!

Theorem: If n is an integer, then $\lceil n/2 \rceil + \lfloor n/2 \rfloor = n$.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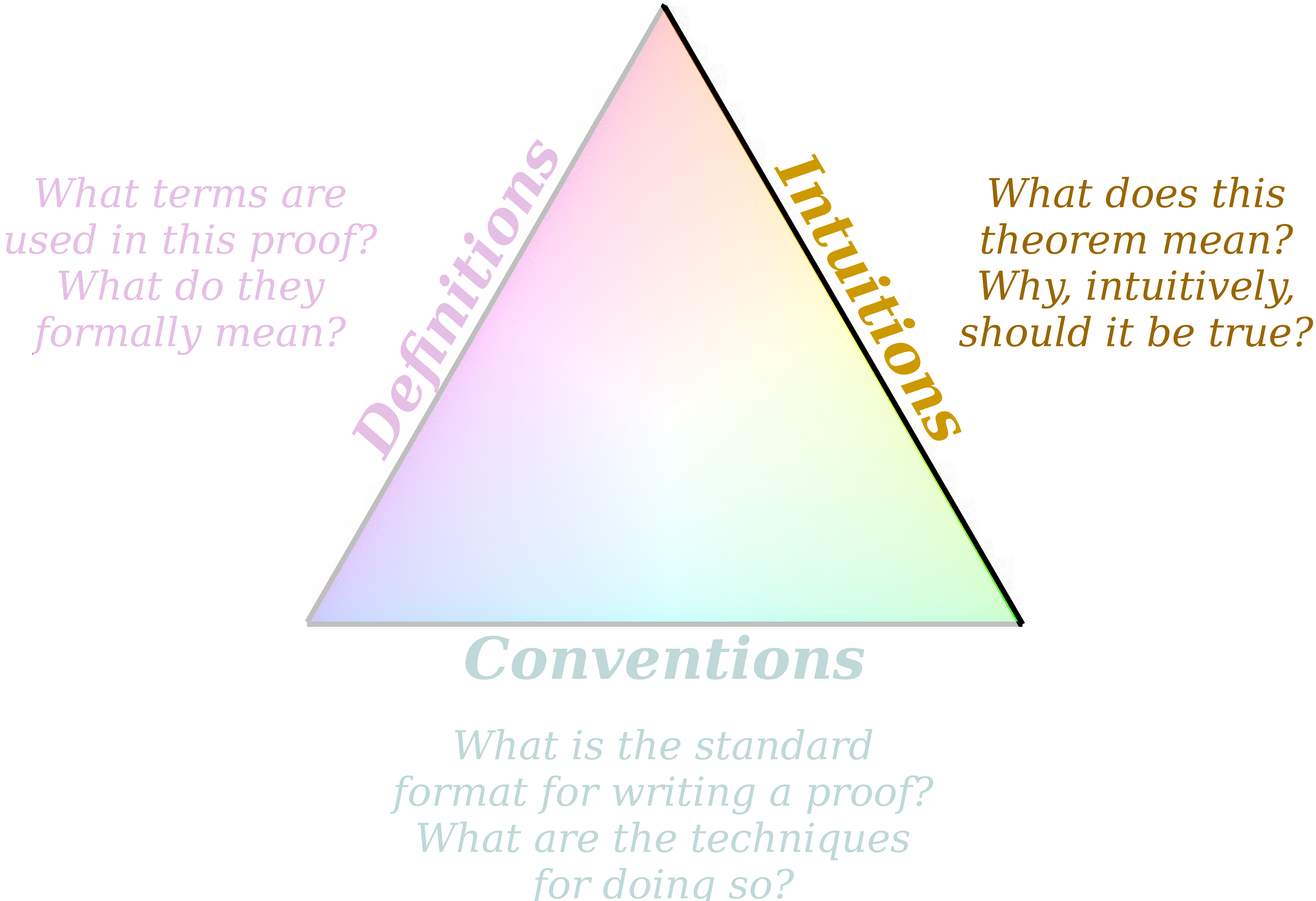
Floors and Ceilings

- The notation $\lceil x \rceil$ represents the ***ceiling*** of x, the smallest integer greater than or equal to x.
  - What is $\lceil 1 \rceil$? What's $\lceil 1.2 \rceil$? What's $\lceil -1.2 \rceil$?
  - ***Intuition:*** Start at x on the number line, then move to the right until you hit a tick mark.
- The notation $\lfloor x \rfloor$ represents is the ***floor*** of x, the largest integer less than or equal to x.
  - What is $\lfloor 1 \rfloor$? What's $\lfloor 1.2 \rfloor$? What's $\lfloor -1.2 \rfloor$?
  - ***Intuition:*** Start at x on the number line, then move to the left until you hit a tick mark.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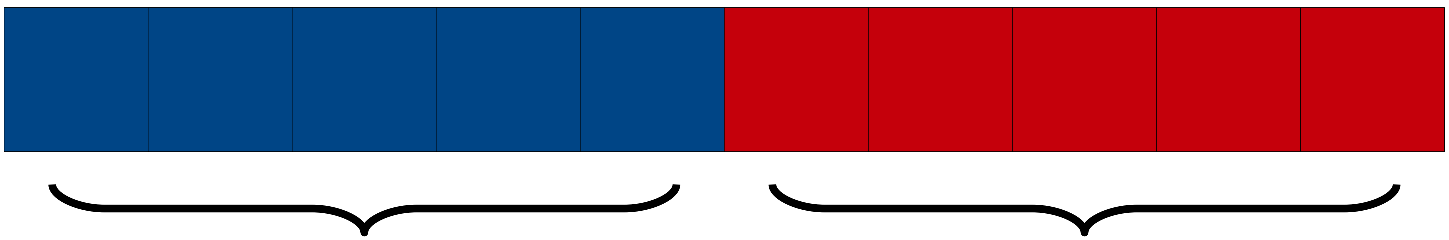
Let’s Draw Some Pictures!



$\lceil n/2 \rceil$ $\lfloor n/2 \rfloor$

$$n = 2k$$

Theorem: If n is an integer, then $\lceil n/2 \rceil + \lfloor n/2 \rfloor = n$.

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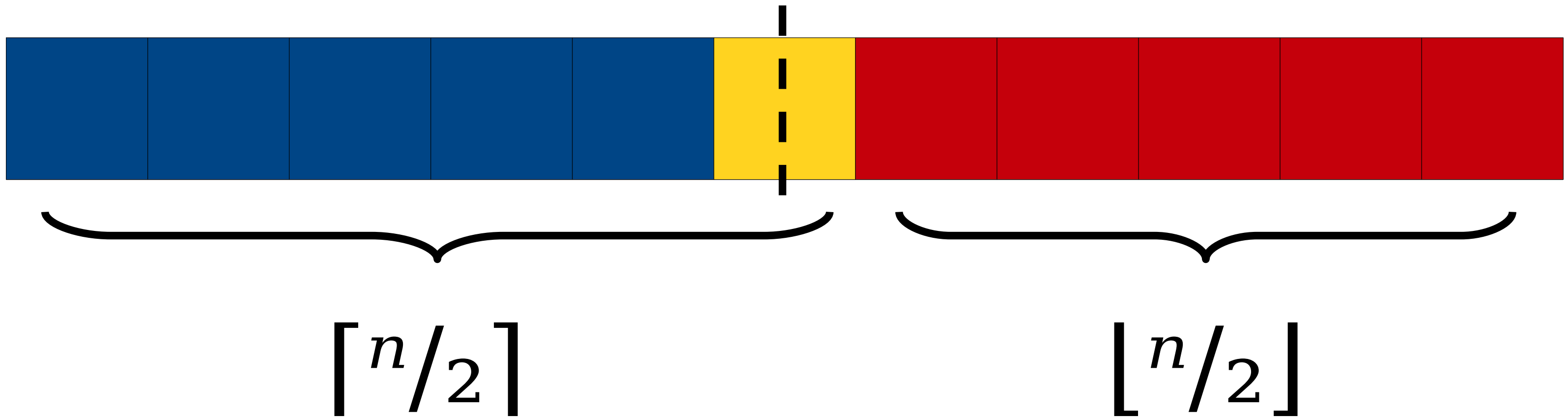

$$n = 2k + 1$$

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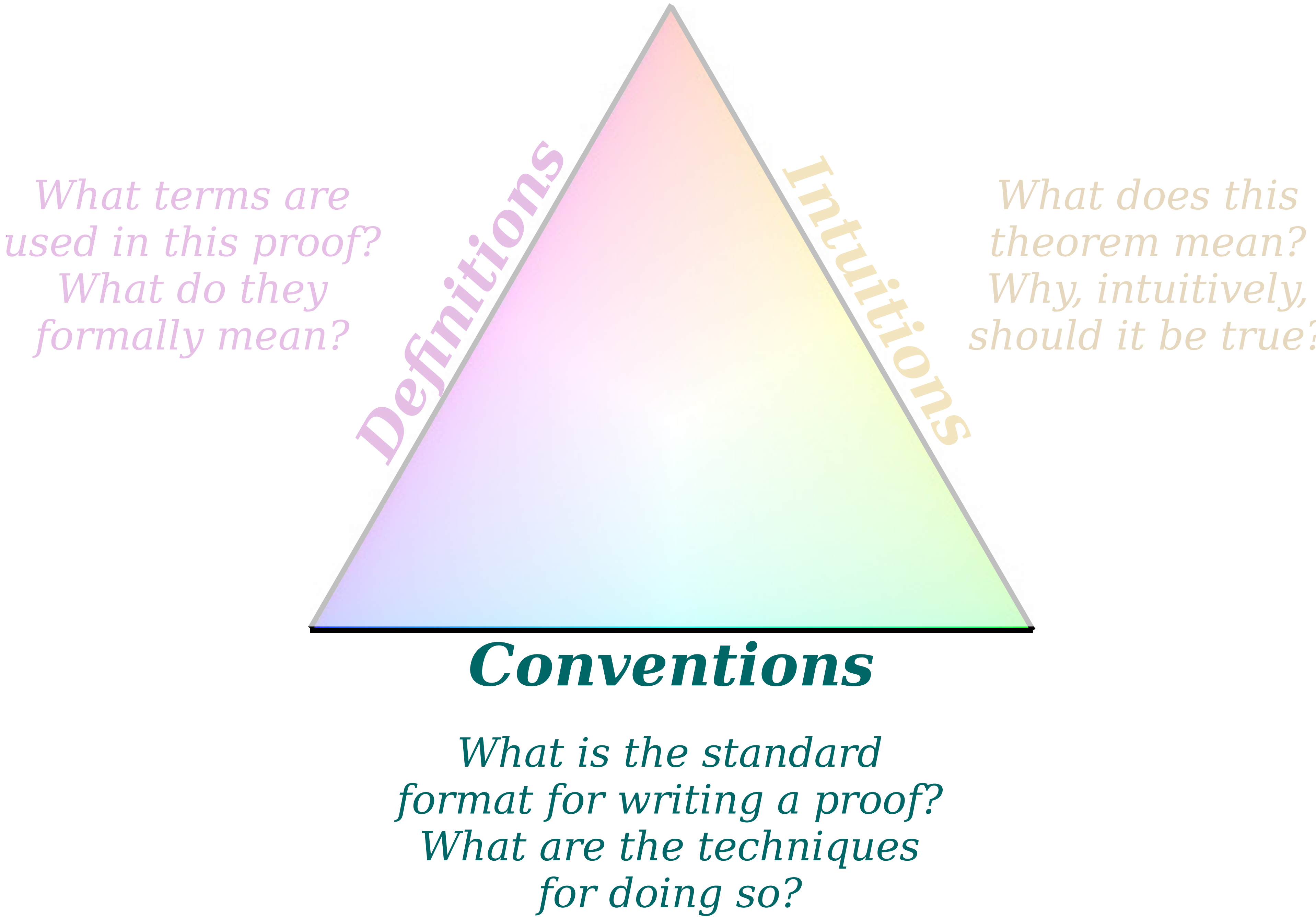
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Theorem: If n is an integer, then $\lfloor n/2 \rfloor + \lceil n/2 \rceil = n$.

Proof: Let n be an integer. We will show that $\lfloor n/2 \rfloor + \lceil n/2 \rceil = n$. To do so, we consider two cases:

Case 1: n is even.

Case 2: n is odd.

Theorem: If n is an integer, then $\lfloor n/2 \rfloor + \lceil n/2 \rceil = n$.

Proof: Let n be an integer. We will show that $\lfloor n/2 \rfloor + \lceil n/2 \rceil = n$. To do so, we consider two cases:

Case 1: n is even.

This is called a *proof by cases* (or *proof by exhaustion*). We split apart into one or more cases and confirm that the result is indeed true in each of them.

(Think of it like an if/else or switch statement.)

Case 2: n is odd.

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Case 1: n is even. This means there is an integer k such that $n = 2k$. Some algebra then tells us that

$$\left\lceil \frac{n}{2} \right\rceil + \left\lfloor \frac{n}{2} \right\rfloor = \left\lceil \frac{2k}{2} \right\rceil + \left\lfloor \frac{2k}{2} \right\rfloor$$

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$$= \lceil k \rceil + \lfloor k \rfloor$$

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Case 1: n is even. This means there is an integer k such that $n = 2k$. Some algebra then tells us that

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In either case, we see that $\lfloor n/2 \rfloor + \lceil n/2 \rceil = n$, as required.

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At the end of a split into cases, it's a nice courtesy to explain to the reader what it was that you established in each case.

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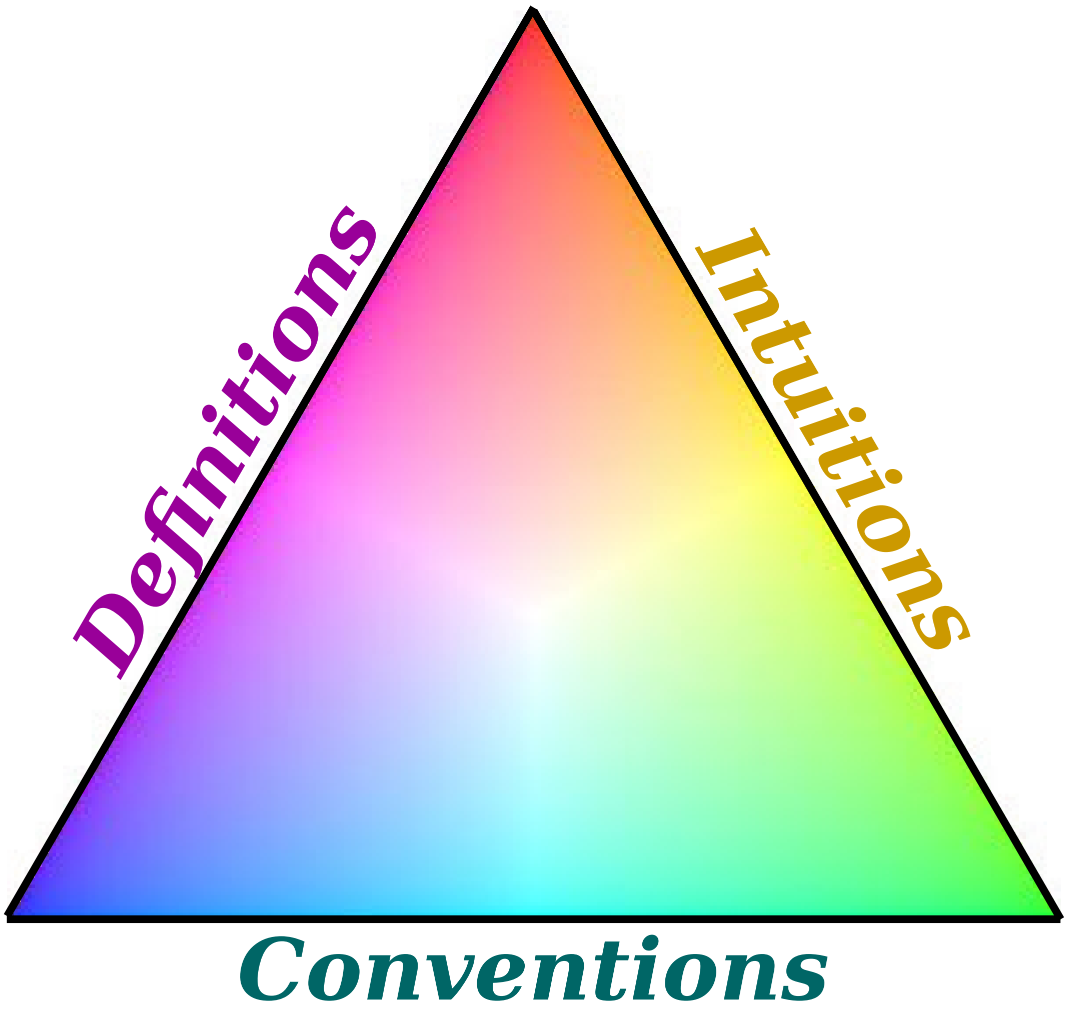
To Recap



Writing a good proof requires a blend of definitions, intuitions, and conventions.

An integer n is ***even*** if there is an integer k where $n = 2k$.

An integer n is ***odd*** if there is an integer k where $n = 2k+1$.

Definitions tell us what we need to do in a proof.
Many proofs directly reference these definitions.

Let’s Draw Some Pictures!

Let’s Do Some Math!

Let’s Try Some Examples!

Building intuition for results requires creativity, trial, and error.

- Prove universal statements by making arbitrary choices.
- Prove existential statements by making concrete choices.
- Prove "If P, then Q" by assuming P and proving Q.
- Write in complete sentences.
- Number sub-formulas when referring to them.
- Summarize what was shown in proofs by cases.
- Articulate your start and end points.

Mathematical proofs have established conventions that increase rigor and readability.

Your Action Items

- ***Read "Guide to ∈ and ⊆," "Guide to Proofs," and "Guide to Partners."***
  - There's a lot of goodies in there.
- ***Finish and submit Problem Set 0.***
  - Don't put this off until the last minute!
- ***(Optionally) Fill out the Problem Set Matchmaker form.***
  - Want us to connect you with someone else? This is a great way to get started.

Next Time

- ***Indirect Proofs***
  - How do you prove something without actually proving it?
- ***Mathematical Implications***
  - What exactly does "if *P*, then *Q*" mean?
- ***Proof by Contrapositive***
  - A helpful technique for proving implications.
- ***Proof by Contradiction***
  - Proving something is true by showing it can't be false.

Appendix: ***Proofs as Dialogs***

Proofs as a Dialog

Let n be an arbitrary odd integer.

Since n is an odd integer, there is an integer k such that $n = 2k + 1$.

Now, let $z = k - 34$.

Proofs as a Dialog

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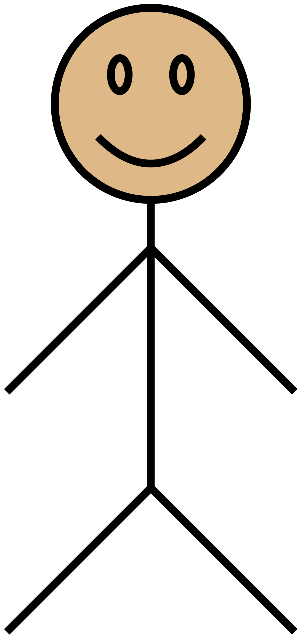

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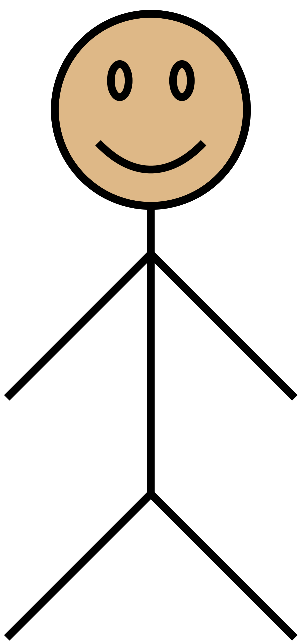

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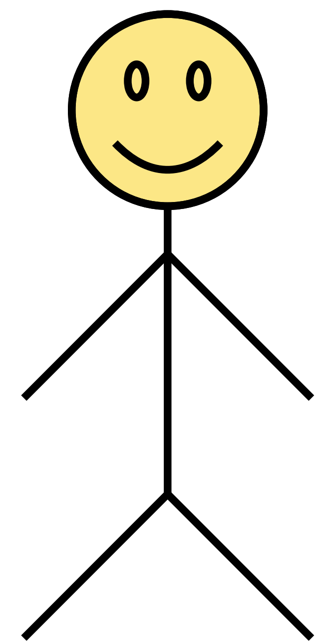

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Since n is an odd integer, there is an integer k such that $n = 2k + 1$.

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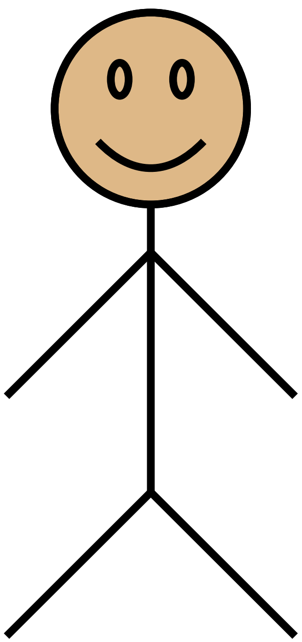
$n = 137$

Reader *Picks*



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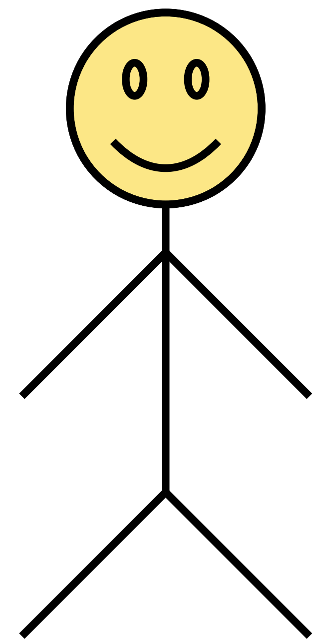

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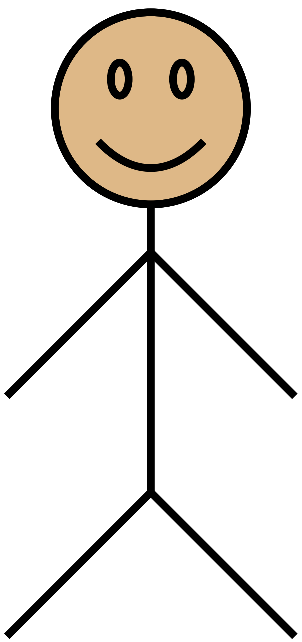

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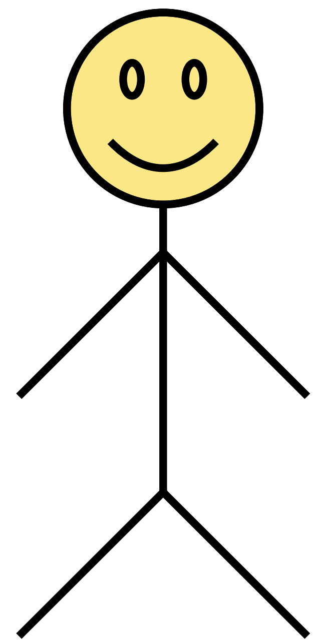

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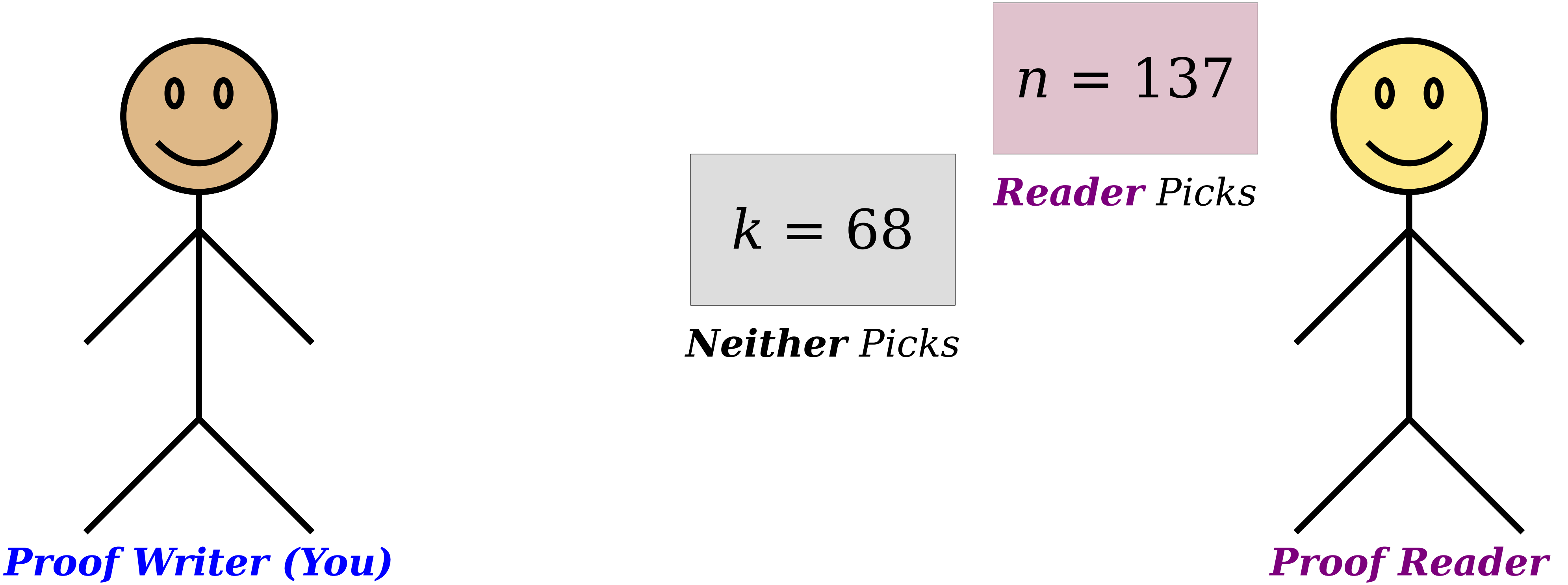

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Let n be an arbitrary odd integer.

Since n is an odd integer, there is an integer k such that $n = 2k + 1$.

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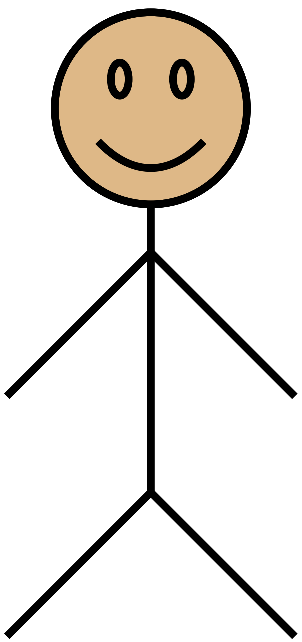
$n = 137$

Reader *Picks*

$k = 68$

Neither *Picks*



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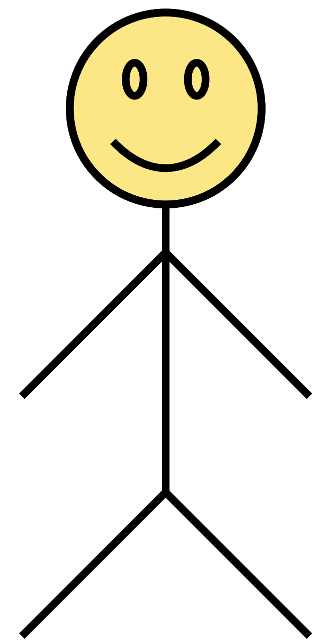

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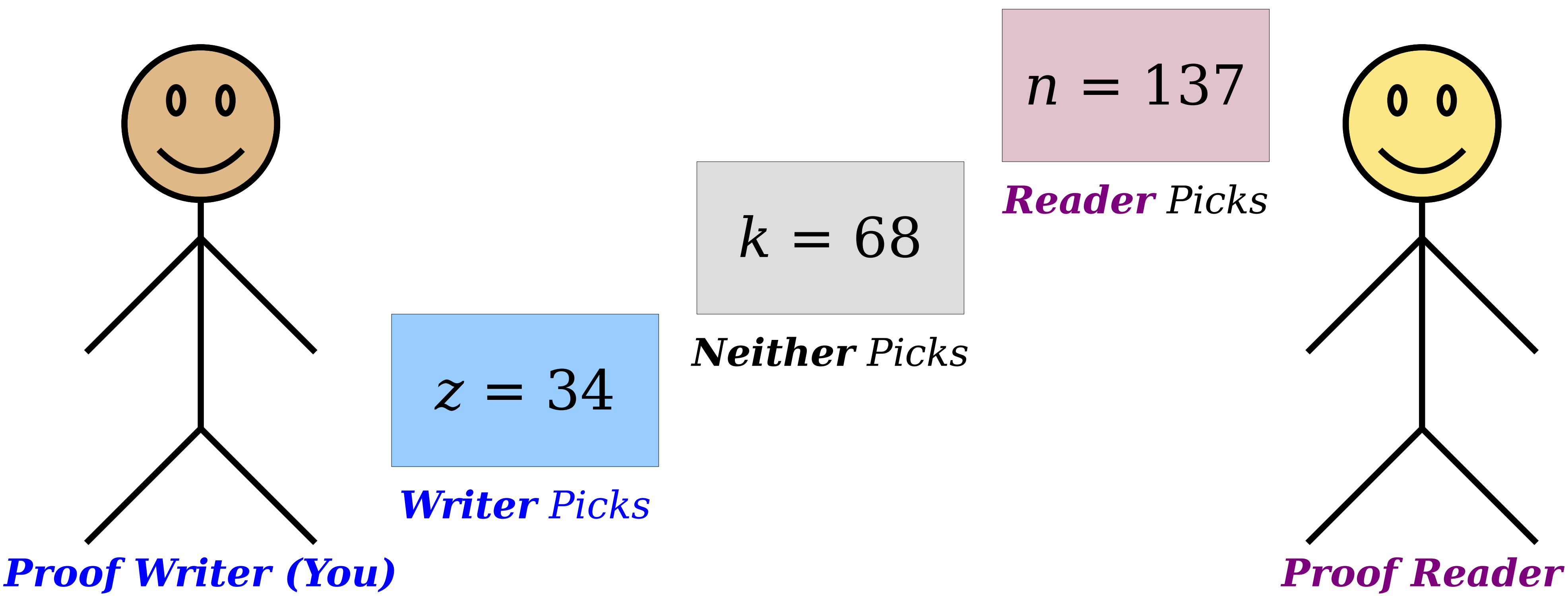

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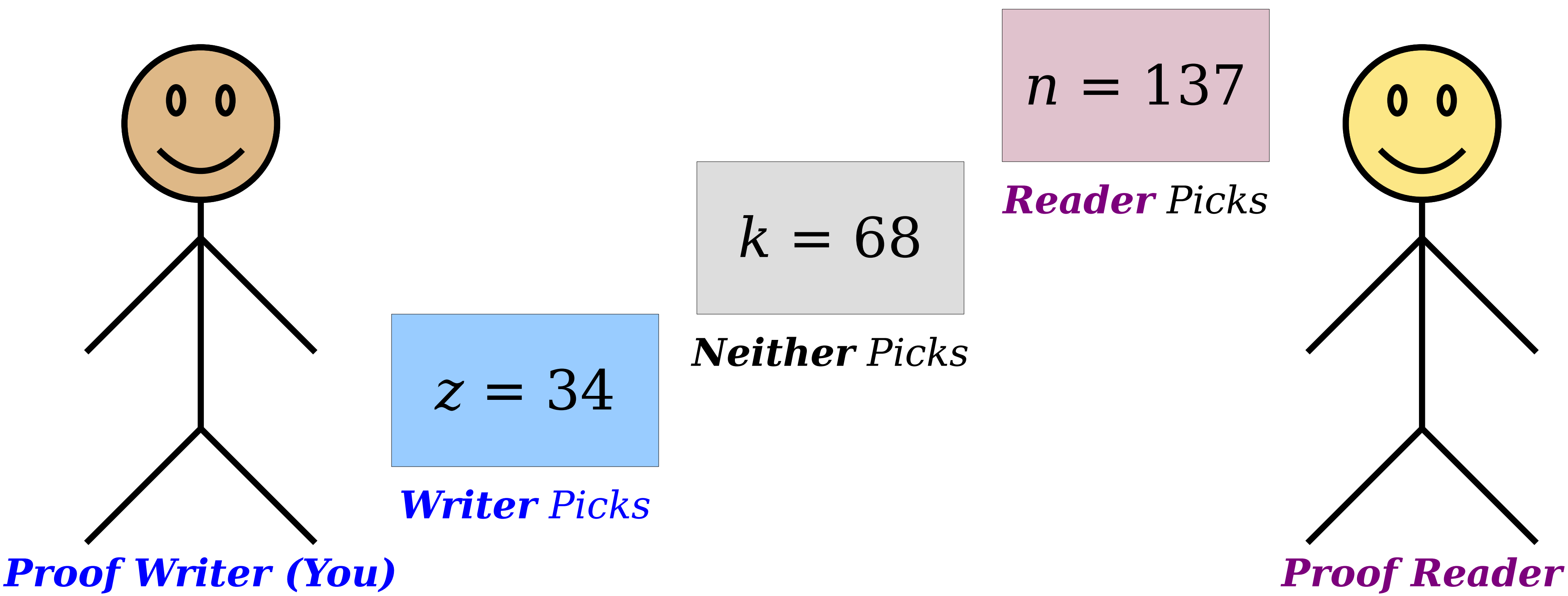

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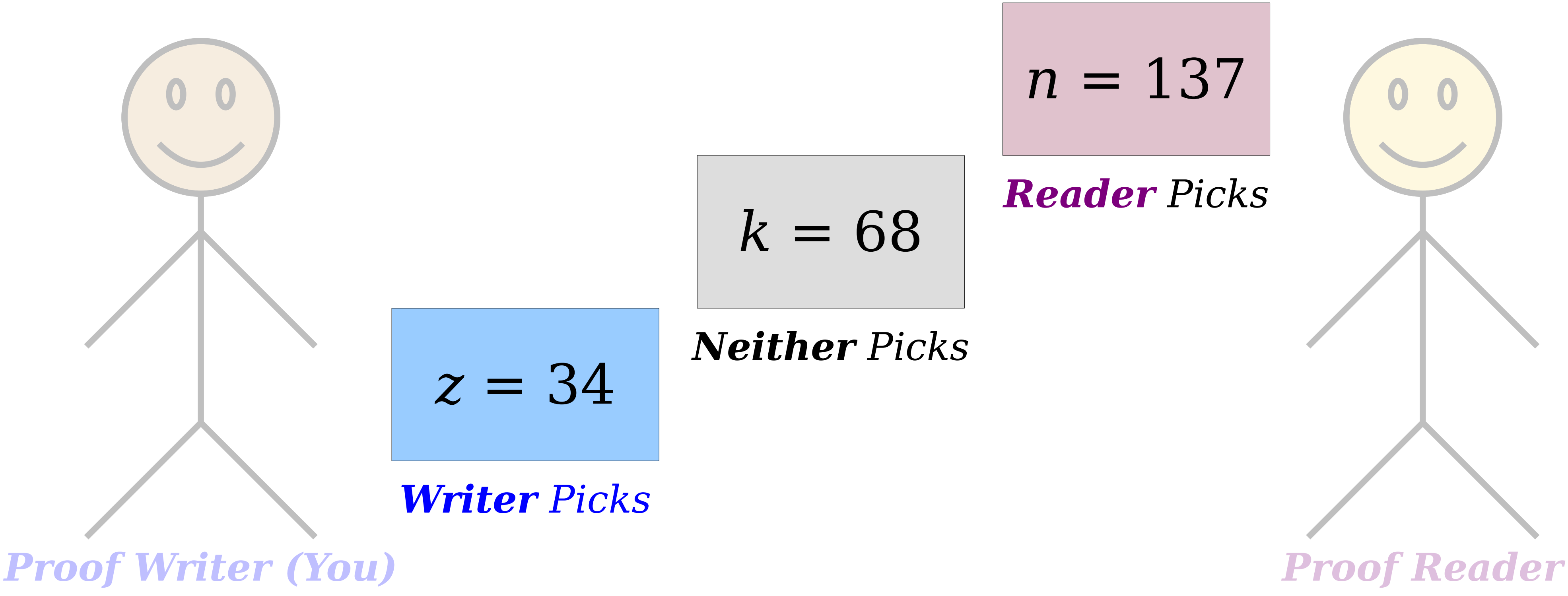


Each of these variables has a distinct, assigned value.

Each variable was either picked by the reader, picked by the writer, or has a value that can be determined from other variables.

Since … r

Now, let $z = k - 34$.

$n = 137$

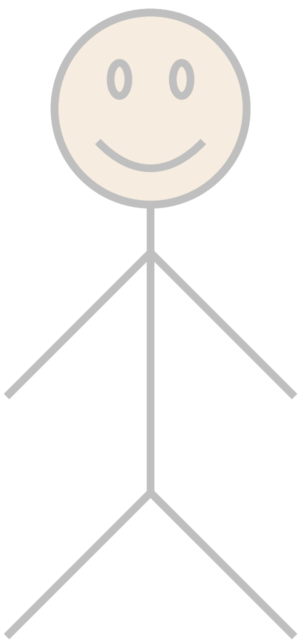
Reader *Picks*

$k = 68$

Neither *Picks*

$z = 34$

Writer *Picks*



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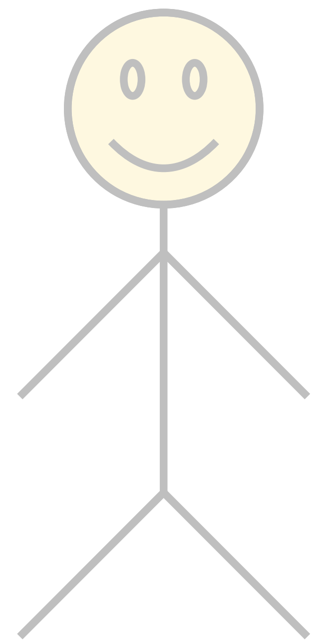


Proof Reader

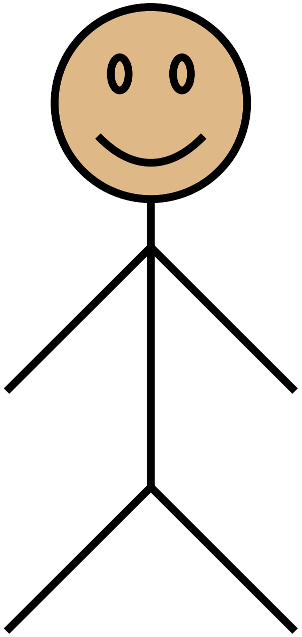
Who Owns What?

- The ***reader*** chooses and owns a value if you use wording like this:
  - Pick a natural number n.
  - Consider some $n \in \mathbb{N}$.
  - Fix a natural number n.
  - Let n be a natural number.
- The ***writer*** (you) chooses and owns a value if you use wording like this:
  - Let $r = n + 1$.
  - Pick $s = n$.
- ***Neither*** of you chooses a value if you use wording like this:
  - Since n is even, we know there is some $k \in \mathbb{Z}$ where $n = 2k$.
  - Because n is odd, there must be some integer k where $n = 2k + 1$.

Proofs as a Dialog

Let x be an arbitrary even integer.

Then for any even x, we know that $x+1$ is odd.



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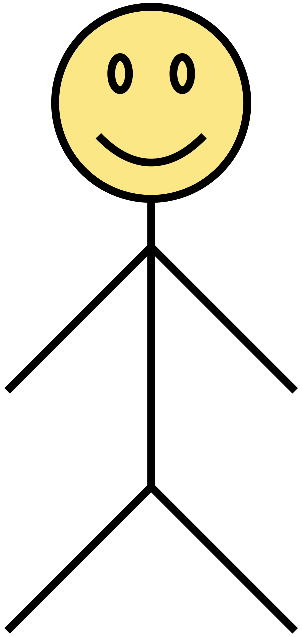

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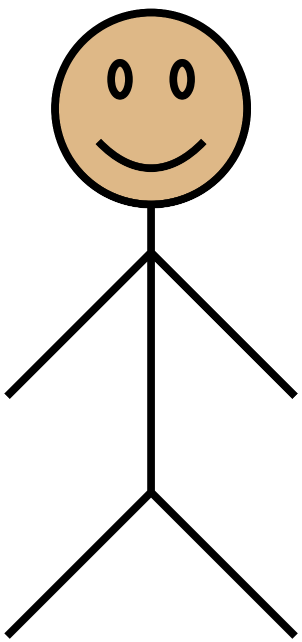

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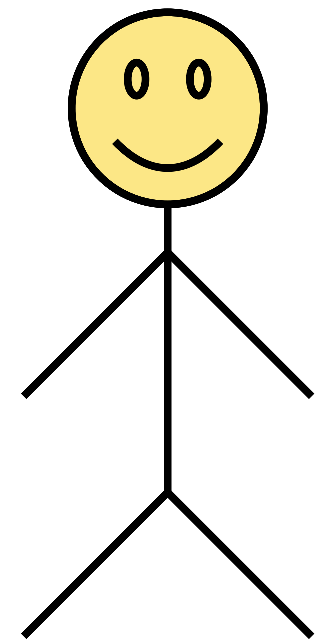

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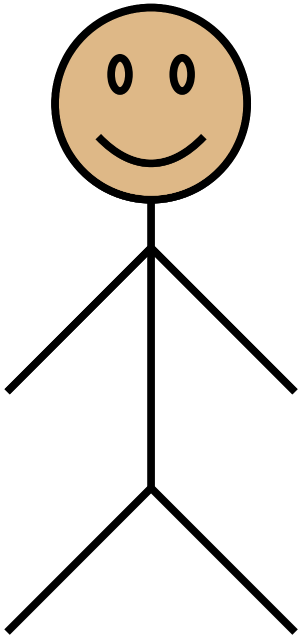
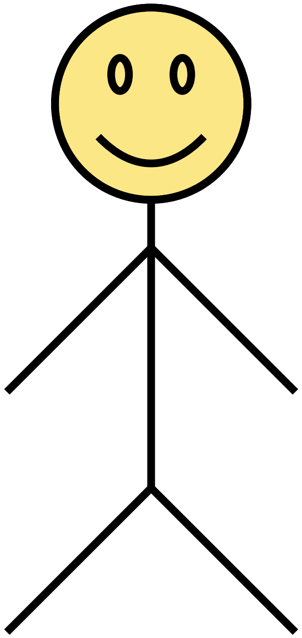


Proof Writer (You)

$x = 242$

Reader *Picks*



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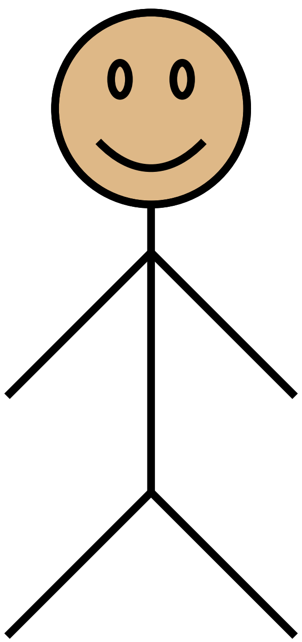

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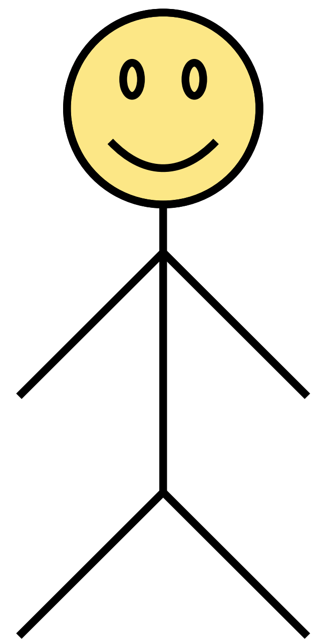

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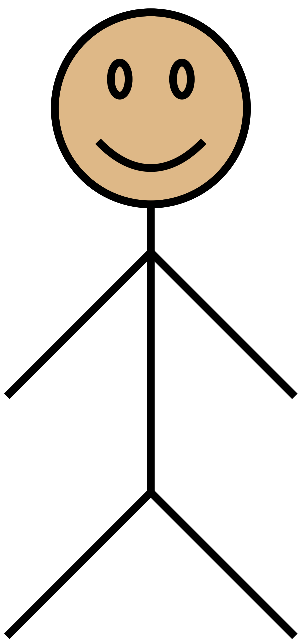

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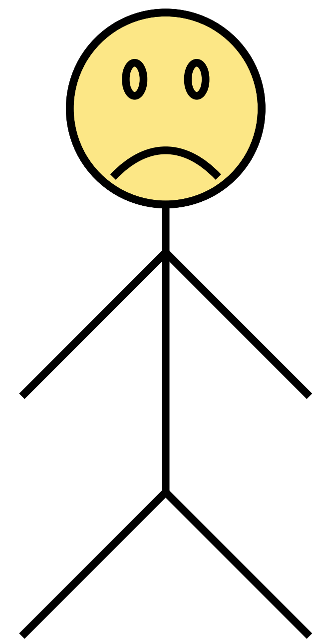

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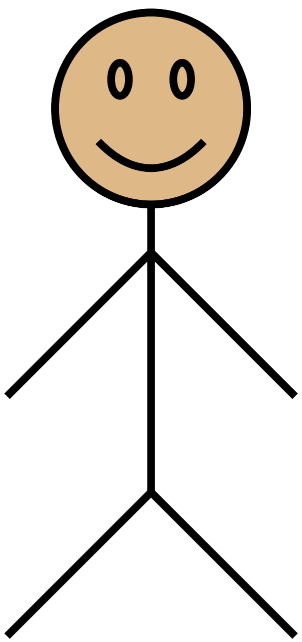

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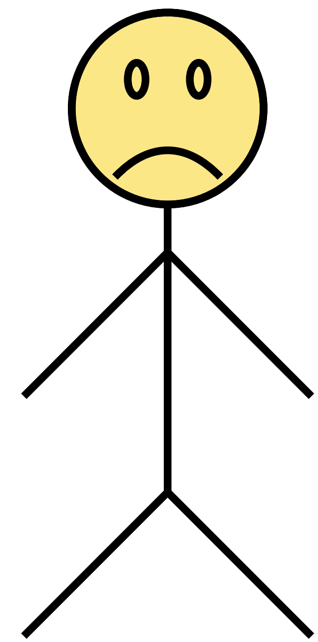

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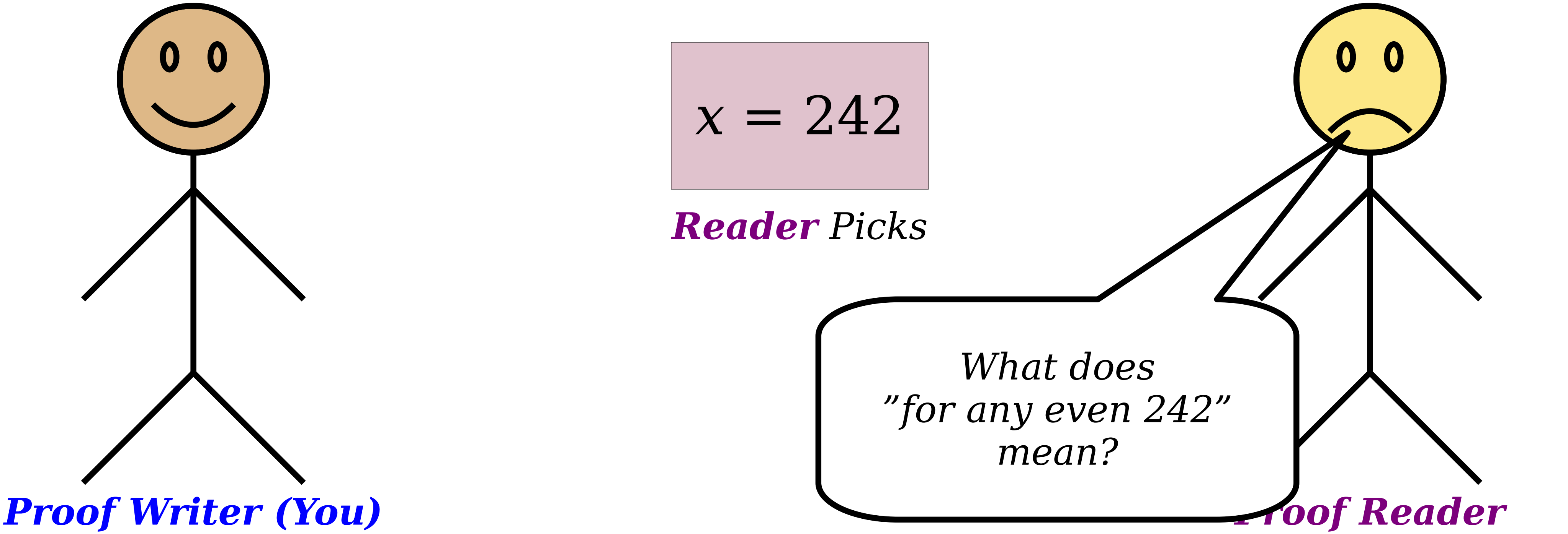

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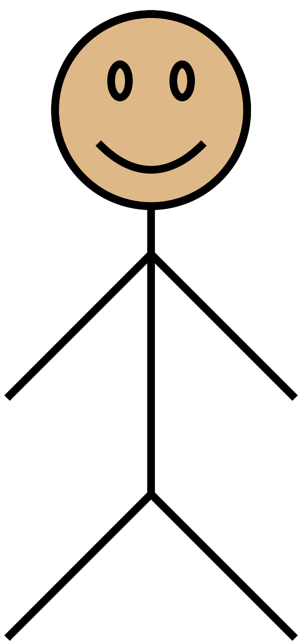
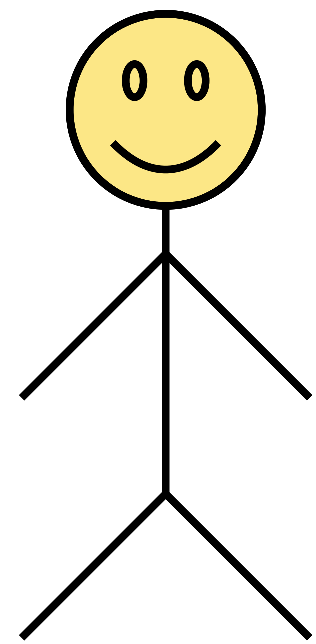
Let x be an arbitrary even integer.

Since x is even, we know that $x+1$ is odd.



Proof Writer (You)



Proof Reader

Proofs as a Dialog

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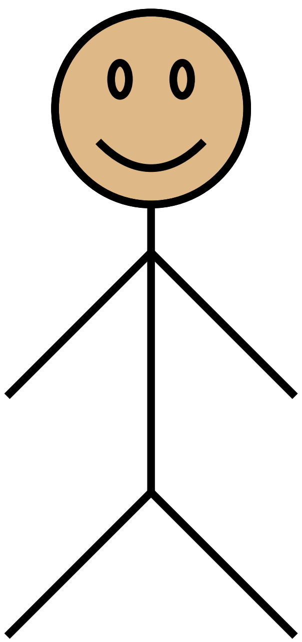
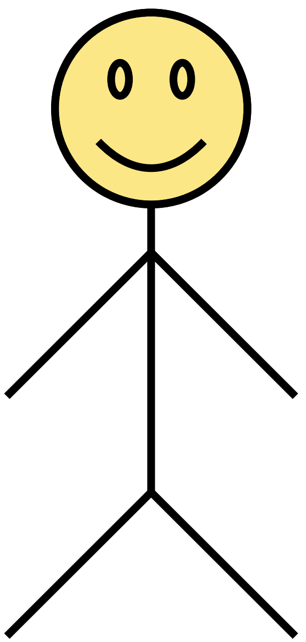


Proof Writer (You)

$x = 242$

Reader *Picks*



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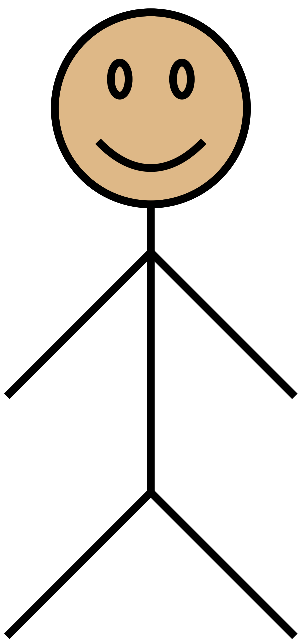

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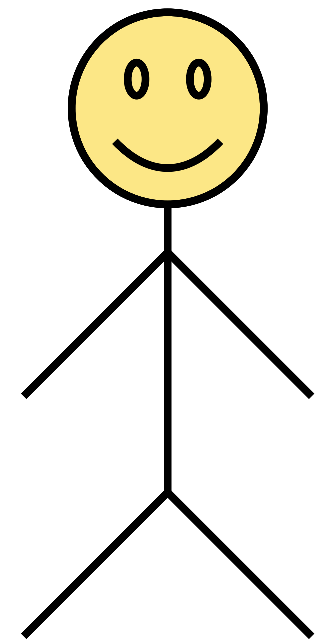


Proof Reader

Every variable needs a value.

Avoid talking about "all x" or "every x" when manipulating something concrete.

To prove something is true for any choice of a value for x, let the reader pick x.

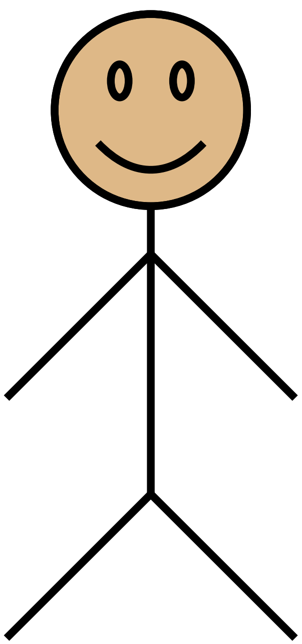
Once you've said something like

Let x be an integer.
Consider an arbitrary $x \in \mathbb{Z}$.
Pick any x.

Do not say things like the following:

This means that ***for any*** $x \in \mathbb{Z}$ …
So ***for all*** $x \in \mathbb{Z}$ …

Proofs as a Dialog



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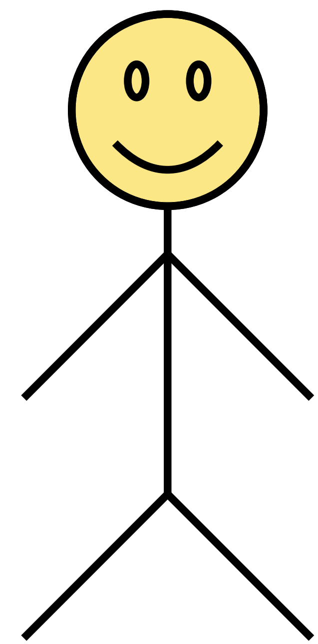
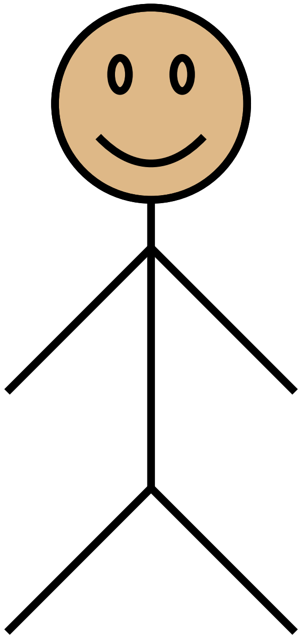


Proof Reader

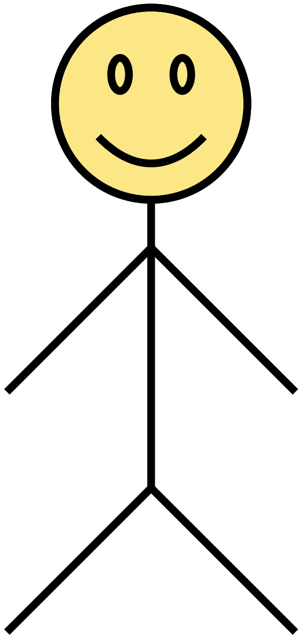
Proofs as a Dialog

Pick two integers m and n where $m+n$ is odd.

Let $n = 1$, which means that $m+1$ is odd.



Proof Writer (You)



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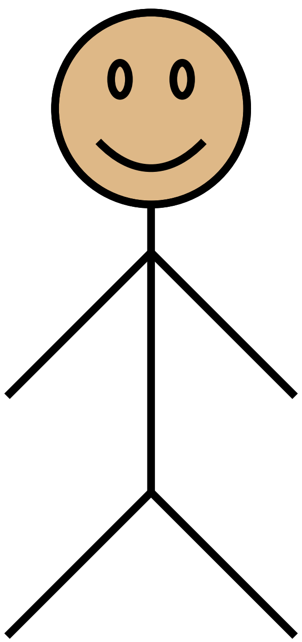

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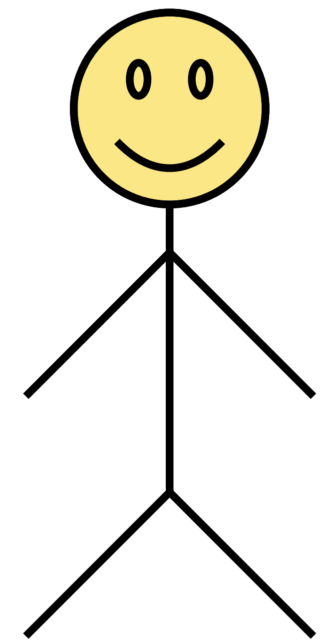

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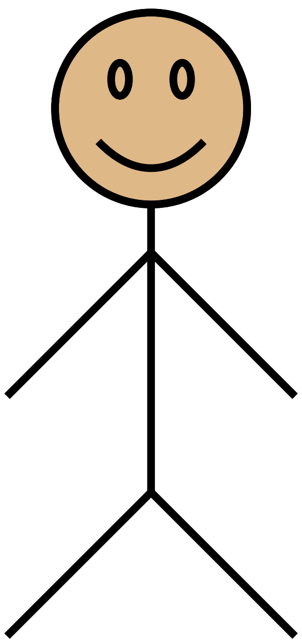


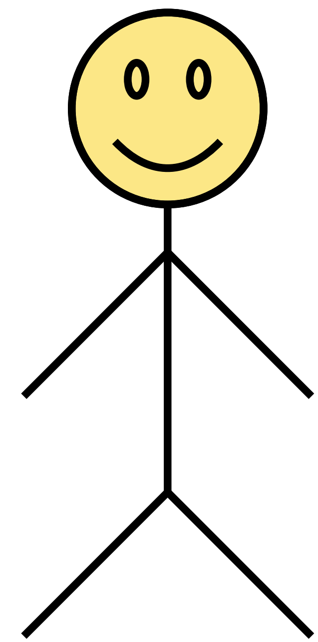
Proof Writer (You)

$m = 103$

Reader *Picks*

$n = 166$

Reader *Picks*



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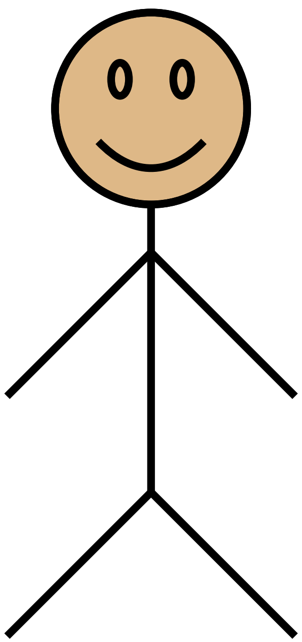

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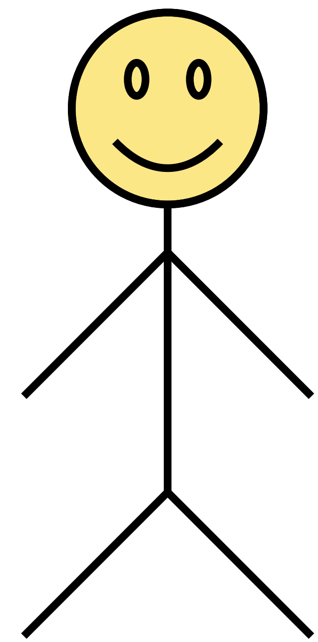

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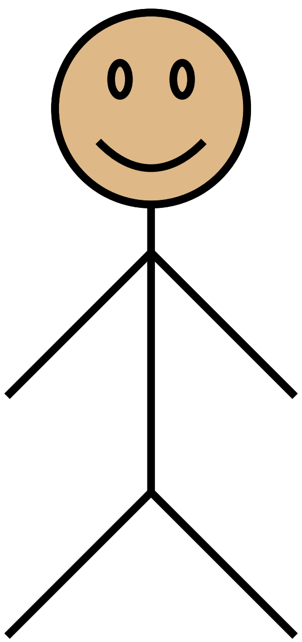

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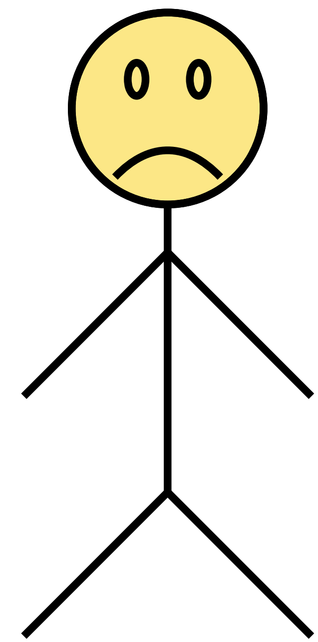

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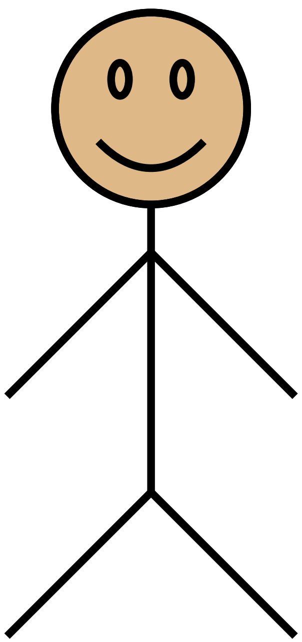

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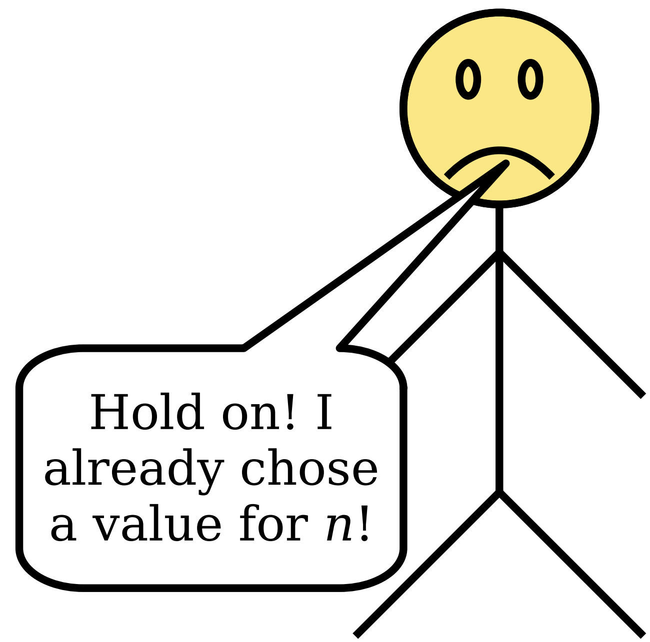


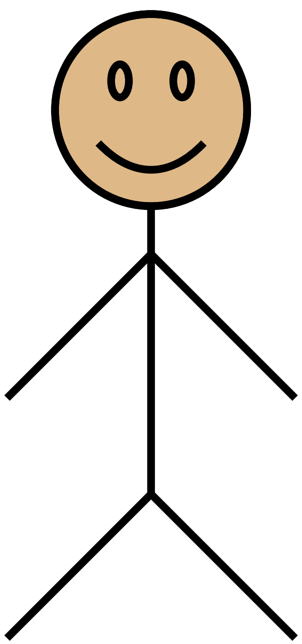
Proof Writer (You)

Proof Reader

Proofs as a Dialog

Let $n = 1$.

Pick any integer m where $m+1$ is odd.



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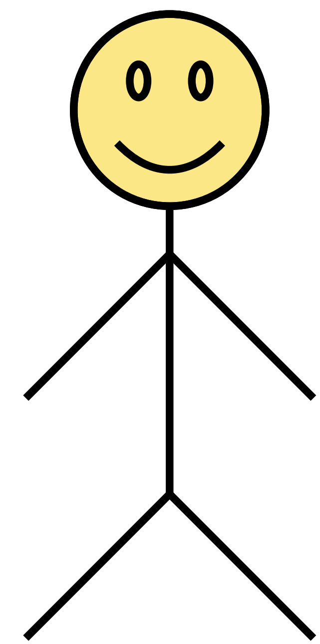

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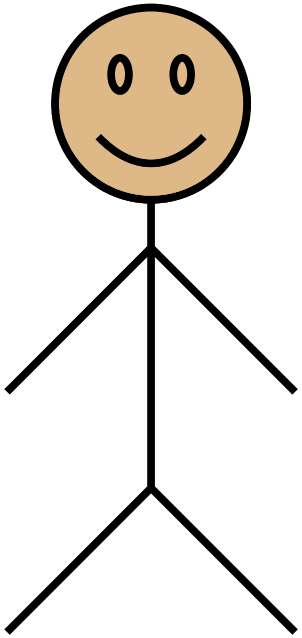

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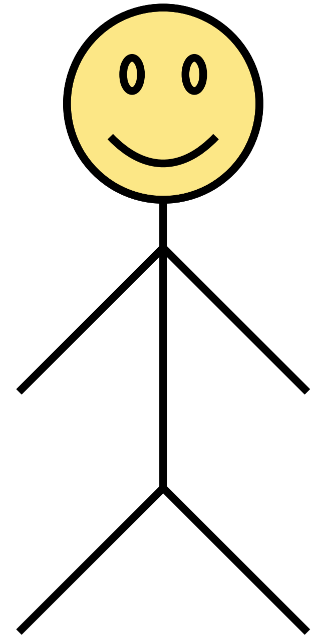

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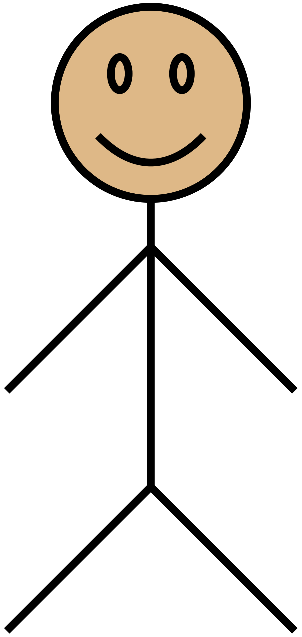
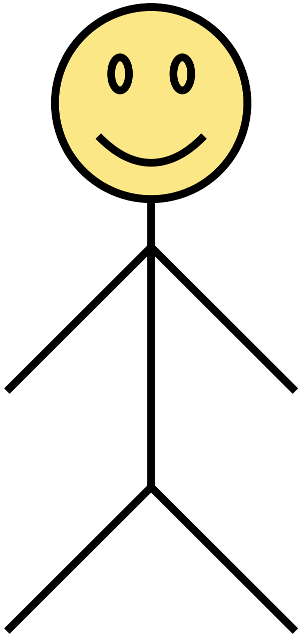


Proof Writer (You)

$n = 1$

Writer *Picks*



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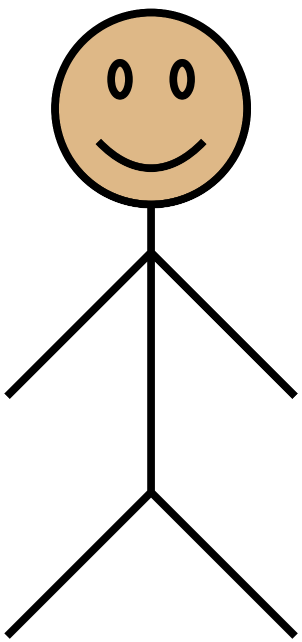

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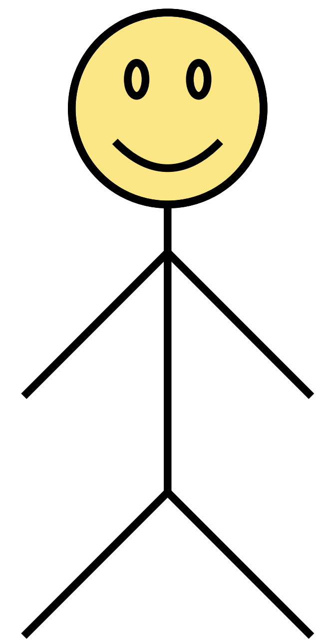
Writer *Picks*



Proof Reader

Proofs as a Dialog

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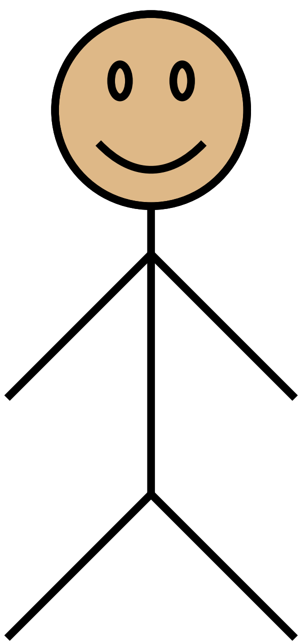
Pick any integer m where $m+1$ is odd.

$m = 166$

Reader *Picks*

$n = 1$

Writer *Picks*



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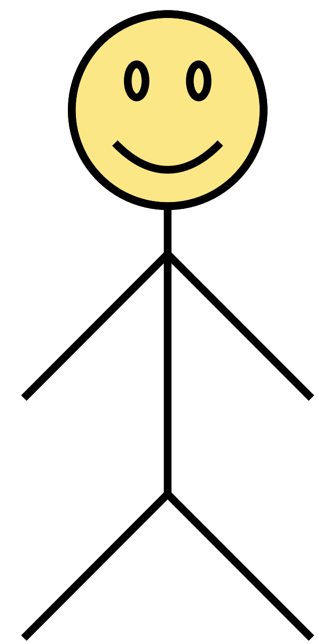

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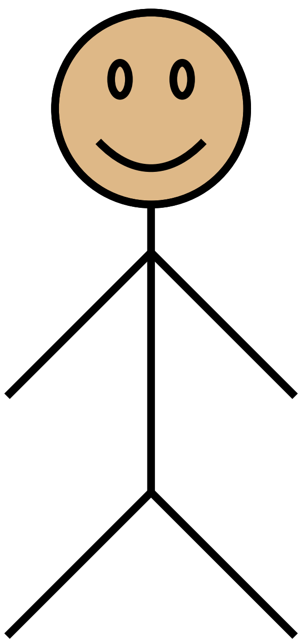
$m = 166$

Reader *Picks*

$n = 1$

Writer *Picks*



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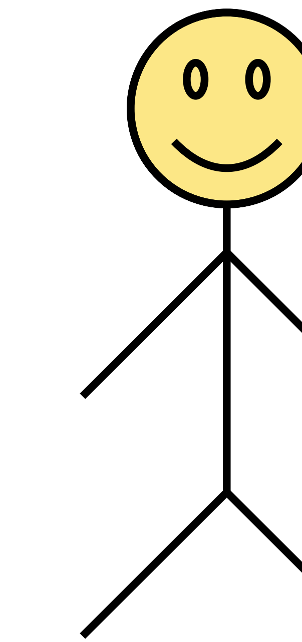


Proof Reader

Proofs as a Dialog

Let $n = 1$.

Pick any integer m where $m+1$ is odd.

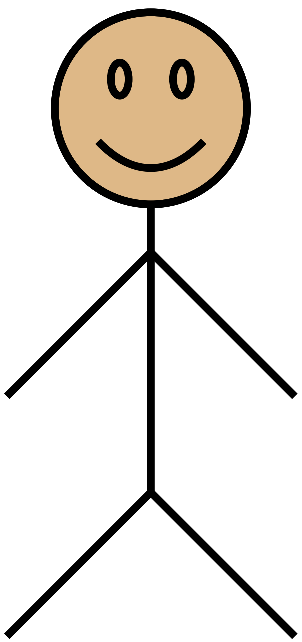
Do we even need n here?

$m = 166$

Reader *Picks*

$n = 1$

Writer *Picks*



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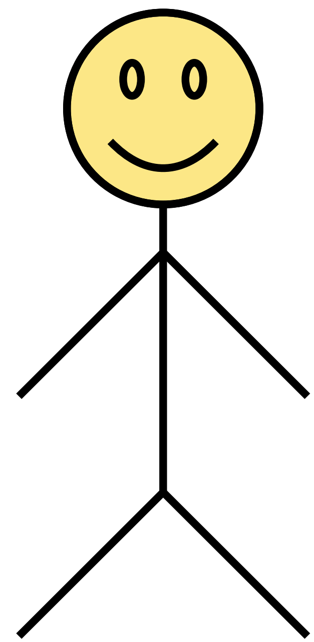

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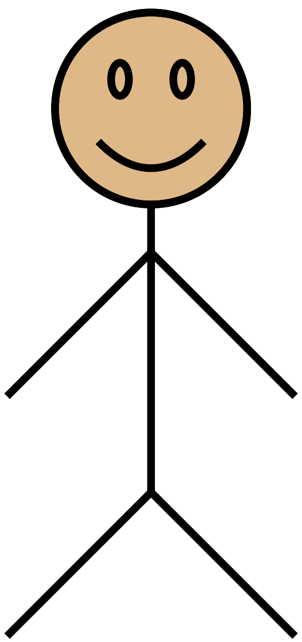

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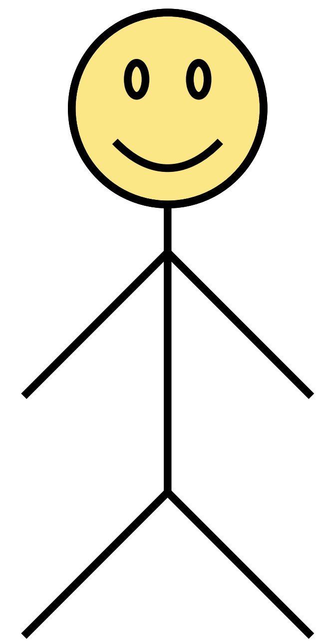

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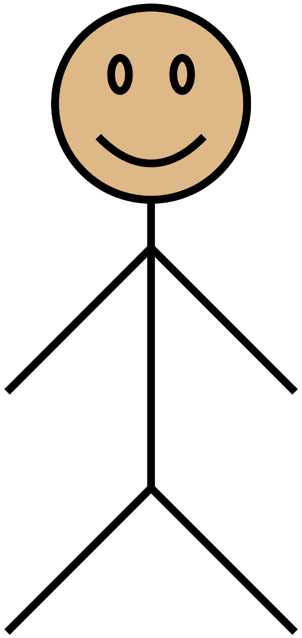

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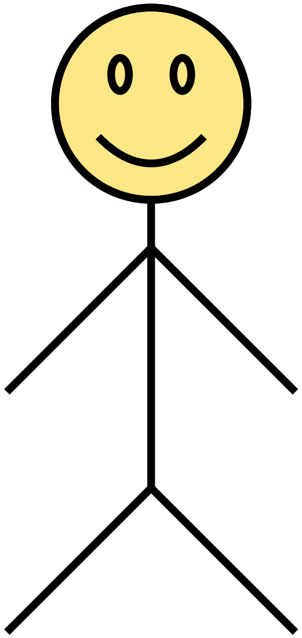

Proof Reader

Be mindful of who owns what variable.

Don’t change something you don’t own.

You don’t always need to name things, especially if they already have a name.